PHYSICAL WAVELETS AND RADAR

A Variational Approach to Remote Sensing*

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Summary

Physical wavelets are acoustic or electromagnetic waves resulting from the emission of a time signal by a localized acoustic or electromagnetic source moving along an arbitrary trajectory in space. Thus they are localized solutions of the wave equation or Maxwell's equations. Under suitable conditions, such wavelets can be used as "basis" functions to construct general acoustic or electromagnetic waves. This gives a *local* alternative to the construction of such waves in terms of (nonlocal) plane waves via Fourier transforms. In this tutorial paper we give a brief, self-contained introduction to physical wavelets and apply them to remote sensing. We define the *ambiguity functional,* a generalization of the radar and sonar ambiguity functions which applies not only to wideband signals but also to targets and radar platforms executing arbitrary *nonlinear* motions.

1. Physical Wavelets

Physical wavelets were introduced in [1–5] and generalized in [6,7]. They are "wavelets" in two distinct senses: In the old sense pioneered by Huygens, meaning localized acoustic or electromagnetic waves [8], and in the modern sense pioneered by Klauder, Morlet, Grossmann, Meyer, Mallat, Daubechies, and others [9–15], meaning functions that are all related by translation and scaling and that form a basis (or "frame") for a whole vector space of functions. In this section we give a brief introduction to such wavelets. For simplicity we specialize to *acoustic* wavelets, which are local scalar solutions of the wave equation in space-time. Electromagnetic wavelets are closely related but somewhat more complicated, since they take into account the vector nature of electric and magnetic fields and the accompanying polarizations; see [5], Chapter 9.

The wavelets described here are more general, and at the same time conceptually simpler, than those developed in [5]. They are generated physically when a point source

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transmitting an *arbitrary* time signal $\psi(t)$ executes an *arbitrary* motion. By contrast, the wavelets in [5] were defined indirectly, by analytically continuing solutions to complex space-time via the *analytic-signal transform.* That forces a virtually unique choice of the transmitted time signal and restricts the velocity of the source to be uniform. As a direct application of the new wavelets, we generalize the idea of ambiguity functions. The usual ambiguity function [16–19] computes the range and velocity of the target by estimating the time delay and Doppler shift of the return through a *time-frequency analysis,* using the outgoing signal as a *basic window.* But it is known that the Doppler effect acts by *scaling* the signal rather than shifting its frequency spectrum. For wideband signals, the Doppler shift is not uniform (it becomes frequency-dependent), and the scale factor must be used to estimate the velocity. The *wideband ambiguity function* [5,20–26] thus depends on the time delay and the Doppler scale factor. It amounts to a *time-scale (wavelet) transform* of the return using the outgoing signal $\psi(t)$ as a *basic wavelet.* However, the target must still be assumed to move uniformly. We show that the new wavelets can be used to estimate *arbitrary* target motion via the *ambiguity functional,* which further generalizes the ambiguity function concept.

To keep the equations uncluttered so that the concepts stand out, we use the following compact notation: The position vector in space $\mathbf{R}^3$ will be denoted by the boldface letter $\mathbf{x}$, and time will be denoted by τ. (The letter t will be used to parameterize trajectories in space.) The point in four-dimensional space-time $\mathbf{R}^4$ with position vector $\mathbf{x}$ and time coordinate τ will be denoted by the italic letter x, representing a "4-vector":

$$x = (\mathbf{x}, \tau)\,.$$

The wave operator in space-time will be denoted by $\Box$. It operates on functions $F(x) = F(\mathbf{x}, \tau)$ by

$$\Box\, F(x) = \frac{1}{c^2}\,\frac{\partial^2}{\partial \tau^2}\,F(\mathbf{x},\tau) - \nabla^2 F(\mathbf{x},\tau)\,,$$

where c is the constant propagation speed of the waves (sound or light) and ∇ is the gradient with respect to the space variables. A given scalar source function $J(\mathbf{x}, \tau)$ distributed in space-time generates an acoustic wave $F(\mathbf{x}, \tau)$ satisfying

$$\Box\, F(x) = J(x)\,.$$

The solution $F(x)$ is given by the four-dimensional convolution integral

$$F(x) = \int d^4x'\; G(x - x')J(x')\,, \tag{1}$$

where G is the *retarded Green function*

$$G(x) = G(\mathbf{x}, \tau) = \frac{\delta(\tau - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|}\;. \tag{2}$$

Thus the effect of $J(\mathbf{x}', \tau')$ is felt at $(\mathbf{x}, \tau)$ if and only if $\tau = \tau' + |\mathbf{x} - \mathbf{x}'|/c$, which means that the waves propagate *causally* at speed c. Suppose now that we have a small transmitter, one that can be regarded as a point source, moving along an arbitrary trajectory in space given by $\mathbf{x} = \mathbf{r}(t)$. While moving along this trajectory, the transmitter is emitting a (real) signal $\psi(t)$. The trajectory of the transmitter in *space-time* $\mathbf{R}^4$ can be parameterized as a 4-vector function

$$\alpha(t) = (\mathbf{r}(t), t).$$

Since the source function is now concentrated along $\alpha(t)$, the resulting wave (1) reduces to

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} dt\ G(x - \alpha(t))\, \psi(t) \\ &= \int_{-\infty}^{\infty} dt\ G(\mathbf{x} - \mathbf{r}(t), \tau - t)\, \psi(t). \end{aligned} \tag{3}$$

Given the trajectory $\alpha(t)$, let us define the *emission operator* E_α as the operator transforming the *time signal* $\psi(t)$ into the *space-time wave* $F(x)$ given by (3):

$$\boxed{(E_\alpha \psi)(x) = \int_{-\infty}^{\infty} dt\ G(x - \alpha(t))\, \psi(t)\,.} \tag{4}$$

E_α takes a function of one variable to a function of four variables. It models the process of emitting a time signal, the result of which is a wave defined in space-time. Note that the right-hand side depends on $\alpha(t)$ as a whole, which is why we write E_α instead of $E_{\alpha(t)}$.

When the trajectory $\alpha(t)$ is *linear*, i.e., when the transmitter moves at a constant velocity through space, then $E_\alpha \psi$ turns out to be a translated and scaled version of $\psi(t)$ [6]. The simplest case occurs when the transmitter is at rest at the origin, so that $\mathbf{r}(t) = \mathbf{0}$. Then

$$(E_\alpha \psi)(\mathbf{x}, \tau) = \frac{\psi(\tau - \rho/c)}{4\pi\rho}\,, \quad \text{where} \quad \rho = |\mathbf{x}|. \tag{5}$$

Since the right-hand side depends only on ρ and τ, wavelets emitted by stationary sources are spherical and can be easily visualized. One such example (for a particular choice of ψ associated with analyticity) is shown in Figure 1.

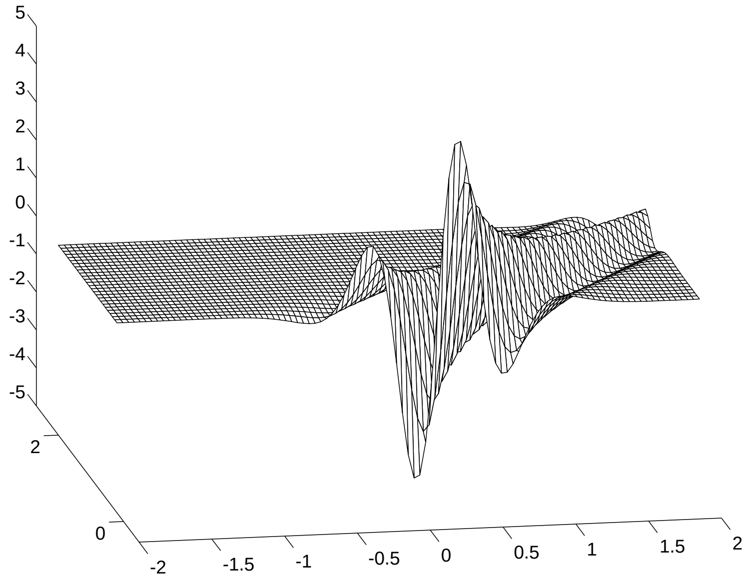

Figure 1. Example of an acoustic wavelet with stationary source as in Equation (5), plotted as a function of time τ (left to right) and radius ρ. This is the emitted part of the wavelet in Figure 11.6 of [5], given by Equation (11.41) with $\alpha = 10$.

Equation (5) states the intuitively obvious fact that the wave received at $\mathbf{x}$ at time τ is a delayed (by ρ/c) and attenuated (by $1/4\pi\rho$) version of the transmitted signal. When the transmitter is moving at a constant nonzero velocity, the resulting wave furthermore exhibits a Doppler effect, which amounts to a *scaling* along the direction of motion. Since translations and scalings form the "raw material" for the standard wavelets [5,12,14,15], we call $E_\alpha \psi$ a *generalized acoustic wavelet.* It is "generalized" because for nonlinear $\alpha(t)$, $E_\alpha \psi$ can no longer be described simply as a translated and scaled version of ψ but is still given by the integral (4). Another property usually required of wavelets is *completeness.* This means that *any* function (belonging to a well-defined vector space of functions, usually a Hilbert space) must be expressible as a linear superposition of wavelets. In our case, where the wavelets must furthermore be solutions of the wave equation, an arbitrary *solution* (belonging to a given vector space of solutions) must be so expressible. Under suitable conditions, a set of emitted physical wavelets of the form $E_\alpha \psi$ with $\alpha(t)$ *linear*, together with a similar set of *absorbed* wavelets, does indeed form such a complete set of solutions. (See [5], Section 9.6 for electromagnetics and Section 11.1 for acoustics.) Finally, wavelets must usually satisfy an *admissibility condition* of the form $\int dt\ \psi(t) = 0$ in order that the associated wavelet transform (defined in Section 3) be invertible. In the case of electromagnetic wavelets, such an admissibility condition follows automatically from the fact that the "DC component" does not propagate; see [5], page 214.

The process of reception works in the opposite direction: A receiver moving along a trajectory $\alpha(t)$ converts an acoustic wave $F(x)$ into a function of time (the received signal) by "measuring" the wave along its trajectory. This can be formulated as a *reception operator* R_α defined by

$$\boxed{(R_\alpha F)(t) = g_\alpha F(\alpha(t))\,,} \tag{6}$$

where g_α is a gain factor associated with the receiver. R_α transforms functions of four variables to functions of one variable. This is only the simplest model for reception, but it will do for the present purpose.

2. Applications to Radar and Sonar

Suppose a transmitter is moving along a trajectory $\alpha(t)$ as above, emitting a time signal $\psi(t)$. The resulting wave $E_\alpha\psi$ is then *reflected* from a point "target" moving along a second trajectory $\beta(t)$, and the reflected wave is finally *received* by a receiver moving along a third trajectory $\gamma(t)$, giving a time signal $\chi(t)$. This situation is depicted schematically in Figure 2.

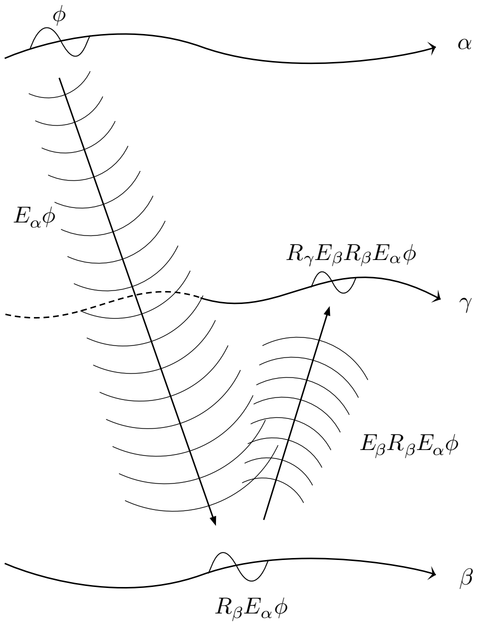

Figure 2. The signal ϕ is emitted from the trajectory α, reflected from β, and received at γ.

General Radar/Sonar problem: Given the trajectories $\alpha(t)$ and $\gamma(t)$ of the transmitter and the receiver, as well as the transmitted and received signals $\psi(t)$ and $\chi(t)$, estimate the trajectory $\beta(t)$ of the target.

Before attempting to solve this problem we must provide a model for the reflection of waves from a point target moving along the trajectory $\beta(t)$. Given an arbitrary wave $F(x)$ in space-time, let us model the reflected wave by assuming that $F(x)$ is first received by the target, then immediately re-emitted. That is, the target momentarily acts as a receiver, then as a transmitter. (In electromagnetics, this corresponds to formulating scattering by observing that the incident wave generates a current on the target, which then radiates the scattered wave.) According to (4) and (6), the reflected wave is then

$$F_{\text{refl}}(x) = (E_\beta R_\beta F)(x) = g_\beta \int_{-\infty}^{\infty} dt \; G(x - \beta(t))F(\beta(t)). \tag{7}$$

Note that reflection is represented by the *operator* $E_\beta R_\beta$ converting a function of four variables (the incident wave) to another function of four variables (the reflected wave). In the present context, the "gain factor" g_β is interpreted as a *reflection coefficient.* With emission, reception, and reflection modeled by (4), (6), and (7), we are ready to tackle the radar/sonar problem. If we knew the target trajectory β (which we do not), then we could compute the received signal as

$$\psi_\beta = R_\gamma E_\beta R_\beta E_\alpha \psi \,, \tag{8}$$

where the dependence on the unknown trajectory β of the target has been made explicit while the dependence on the known trajectories α and γ of the transmitter and receiver is left implicit. Inserting the definitions of the emission and reception operators gives the detailed formula

$$\psi_\beta(t) = g_\gamma g_\beta \iint dt' \, dt'' \; G(\gamma(t) - \beta(t'))G(\beta(t') - \alpha(t''))\psi(t''). \tag{9}$$

Our strategy for solving the radar/sonar problem is to compute ψ_β for a *trial trajectory* β, then *match* the result with the actual received signal $\chi(t)$. "Matching" here means simply taking the *inner product,* i.e., integrating over time:

$$\langle\, \psi_\beta \,, \chi \,\rangle = \int_{-\infty}^{\infty} dt \; \psi_\beta(t)\, \chi(t). \tag{10}$$

Equation (9) shows that the right-hand side of (10) depends on the entire *history* of the target motion.* Hence $\langle\, \psi_\beta \,, \chi \,\rangle$ depends on β as a *functional* rather than an ordinary function. We call it the *ambiguity functional* of β and denote it by $\tilde{\chi}[\beta]$:

$$\tilde{\chi}[\beta] = \langle\, \psi_\beta \,, \chi \,\rangle. \tag{11}$$

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* To be precise, it depends on those values $\beta(t')$ for which (A) there are times t and t'' such that $\chi(t) \neq 0$ and $\psi(t'') \neq 0$, and (B) $\alpha(t'')$, $\beta(t')$ and $\gamma(t)$ are *causally ordered,* so that an acoustic or electromagnetic wave can travel from $\alpha(t'')$ to $\beta(t')$ to $\gamma(t)$ at the speed c. This causality is built into the Green functions in (9).

The *Schwarz inequality* states that

$$|\langle\, \psi_\beta \,, \chi \,\rangle| \le \|\psi_\beta\|\, \|\chi\|, \tag{12}$$

where $\|\psi_\beta\|$ and $\|\chi\|$ are the *norms* of ψ_β and χ, i.e., their squares are the energies:

$$\|\psi_\beta\|^2 = \int_{-\infty}^{\infty} dt\ \psi_\beta(t)^2 \quad \text{and} \quad \|\chi\|^2 = \int_{-\infty}^{\infty} dt\ \chi(t)^2.$$

Furthermore, equality is attained in (12) if and only if the two functions are proportional:

$$|\langle\, \psi_\beta \,, \chi \,\rangle| = \|\psi_\beta\|\, \|\chi\| \iff \chi(t) = k\, \psi_\beta(t) \ \text{ for some constant } \ k. \tag{13}$$

Thus we might try to estimate the actual target trajectory by maximizing the absolute value of $\tilde{\chi}[\beta]$. However, the norm $\|\psi_\beta\|$ depends on β in general, so we cannot just look for a global maximum of $|\tilde{\chi}[\beta]|$. Instead, we can maximize $|\langle\, \psi_\beta \,, \chi \,\rangle|/\|\psi_\beta\|$. A preferable method is to define a new functional

$$\mathcal{E}[\beta] = 1 - \frac{|\langle\, \psi_\beta \,, \chi \,\rangle|}{\|\psi_\beta\|\, \|\chi\|}\ , \tag{14}$$

which we call the *error functional.* Then (12) and (13) state that

$$0 \le \mathcal{E}[\beta] \le 1 \quad \text{and} \quad \mathcal{E}[\beta] = 0 \iff \chi(t) = k\, \psi_\beta(t). \tag{15}$$

> The error functional is nonnegative, and it vanishes if and only if the "trial return" $\psi_\beta(t)$ is indistinguishable (up to a constant factor) from the measured return $\chi(t)$.

Thus, to estimate the target trajectory, we need to minimize $\mathcal{E}[\beta]$. But two qualifications must be added here.

(A) We have not actually given a method for *finding* β so as to minimize $\mathcal{E}[\beta]$. This is a variational problem which must be analyzed separately. One strategy is to compute $\mathcal{E}[\beta]$ in parallel for a set of "likely suspects" β and then declare the trajectory with the least error to be the best estimate in the set. Furthermore, the *value* $\mathcal{E}[\beta]$ can be viewed as an estimate for the error, giving a rough idea of how "close" β is to the actual trajectory. For example, the trajectory of an accelerating target might be parameterized as

$$\beta(t) = (\mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2, t),$$

where $\mathbf{r}_0\,, \mathbf{v}_0$ and $\mathbf{a}$ represent the unknown initial position, initial velocity, and acceleration (assumed uniform) of the target. Then $\mathcal{E}[\beta]$ reduces to an ordinary *function* $\mathcal{E}(\mathbf{r}_0, \mathbf{v}_0, \mathbf{a})$ of the target parameters, which must now be minimized. Of course, if the target's acceleration is actually nonuniform, then none of the above trajectories may have zero error. The one with the least error simply gives the best quadratic approximation.

(B) Even if we find a trajectory β for which $\mathcal{E}[\beta] = 0$, this does not guarantee that $\beta(t)$ is the actual target trajectory since $\mathcal{E}[\beta]$ *is not one-to-one in general.* In other words, the return may not determine the target trajectory uniquely. But clearly we can ask for no more than to find trial trajectories whose returns $\psi_\beta(t)$ are indistinguishable from the measured return. To the extent that $\mathcal{E}[\beta]$ is not one-to-one, the radar/sonar problem is therefore *ambiguous.* The extent of that ambiguity in the present general context needs to be studied.

3. Relation to Ambiguity Functions

We now show that our "ambiguity functional" $\tilde{\chi}[\beta]$ is a generalization of the usual radar ambiguity functions. In most of the literature (for example, [16–19]), ambiguity functions depend on time and frequency. These are *windowed Fourier transforms* of the return and, as explained below, they apply only in *narrowband* situations. A more recent class of ambiguity functions depend on time and scale [5,20–26]. These are *wavelet transforms* of the return, and they apply to *wideband* as well as narrowband situations. But in both cases, the target is assumed to move at a uniform velocity. Hence neither the wideband nor the narrowband ambiguity functions do easily yield information about acceleration. By making a series of restrictive assumptions about the motions of the transmitter, the receiver, and the target, we will see that $\tilde{\chi}[\beta]$ indeed reduces to the usual wideband ambiguity function. Moreover, if the outgoing signal $\psi(t)$ is narrowband and the target does not move too rapidly, then the wideband ambiguity function further reduces to the narrowband ambiguity function. This establishes our claim that $\tilde{\chi}[\beta]$ is a far-reaching generalization of the concept of ambiguity functions.

The assumptions leading to the first reduction are as follows:

A. Monostatic radar:

Assume that the transmitter and the receiver are one and the same, and that both are at rest at the origin in space. That is,

$$\alpha(t) = \gamma(t) = (\mathbf{0}, t). \tag{16}$$

B. Uniform target motion:

Assume that the target moves at a constant velocity $\mathbf{v}$, so that $\beta(t)$ has the form

$$\beta(t) = (\mathbf{r}_0 + \mathbf{v}t, t). \tag{17}$$

Our objective is to find the initial position $\mathbf{r}_0$ and the velocity $\mathbf{v}$. But in practice, targets are *tracked* with a narrow-beam antenna, so the *direction* of $\mathbf{r}_0$ (i.e., the unit vector $\mathbf{n} = \mathbf{r}_0/|\mathbf{r}_0|$) and the transversal ("crossrange") component of $\mathbf{v}$ can be estimated separately from the antenna motion. We therefore need only to find the *range* $r_0 = |\mathbf{r}_0|$ and the radial velocity v, also called the *range rate.* Effectively, we can therefore make the following assumption.

C. No transversal motion (tracking mode):
The target moves directly toward or away from the radar site, i.e.,

$$\mathbf{v} = v\mathbf{n}, \quad \text{where} \quad \mathbf{n} = \frac{\mathbf{r}_0}{|\mathbf{r}_0|} = \frac{\mathbf{r}_0}{r_0} \,.$$

Thus

$$\beta(t) = (r(t)\mathbf{n}, t), \quad \text{where} \quad r(t) = r_0 + vt. \tag{18}$$

Under assumptions $A - C$, our objective is reduced to estimating the range r_0 and the range rate v. When these assumptions are introduced into Equation (9), a computation (performed in the Appendix) gives the simple result

$$\boxed{\psi_\beta(t) = A(t)\psi(\sigma t - \tau)\,,} \tag{19}$$

where

$$\begin{aligned} A(t) &= \frac{g_\gamma g_\beta\,(1 + v/c)}{16\pi^2 r(t)^2} \;=\; \text{attenuation factor} \\ \sigma &= \frac{c - v}{c + v} \;=\; \text{time scaling factor} \\ \tau &= \frac{2r_0}{c + v} \;=\; \text{time shift .} \end{aligned} \tag{20}$$

Each quantity in (20) has a simple intuitive explanation:

1. Equation (19) states that the *instantaneous power* $\psi_\beta(t)^2$ of the return is inversely proportional to $r(t)^4$, in agreement with the *radar range equation* [17–19]. Hence the attenuation factor $A(t)$ is reasonable.

2. The scale factor σ represents the *Doppler effect.* If $0 < v < c$, then $0 < \sigma < 1$ and (19) states that the return is a *stretched* version of $\psi(t)$. If $-c < v < 0$, then $\sigma > 1$ and the return is a *compressed* version of $\psi(t)$. (The relation to the usual Doppler *frequency shift* is discussed below.)

3. Writing (19) as $\psi_\beta(t) = A(t)\psi\big(\sigma(t - \tau')\big)$, the quantity $\tau' = \tau/\sigma = 2r_0/(c - v)$ is interpreted as the *time delay* at $t = 0$ since the signal must travel a round-trip distance of $2r_0$ at the relative velocity $c - v$.

Equation (19) shows that under the assumptions $A - C$, our ambiguity functional reduces to an ordinary *function* of the scale factor σ and the time shift τ :

$$\tilde{\chi}[\beta] \to \tilde{\chi}(\sigma, \tau) \equiv \int_{-\infty}^{\infty} dt\; A(t)\,\psi(\sigma t - \tau)\,\chi(t). \tag{21}$$

In practice it can usually be assumed that $A(t)$ is approximately constant over the time interval where $\psi(\sigma t-\tau)\neq 0$, since $\psi(t)$ is time-limited (usually a *pulse* or a pulse train) and the range $r(t)$ does not change significantly while the reflection takes place ($r_0 \gg |vt|$ for all relevant values of t). In that case, (19) reduces to

$$\psi_\beta(t) \to A\,\psi_{\sigma,\tau}(t), \tag{22}$$

where

$$A = \frac{g_\gamma g_\beta\,(1+v/c)}{16\pi^2 r_0^2}$$

and $\psi_{\sigma,\tau}(t)$ is defined by

$$\boxed{\psi_{\sigma,\tau}(t) = \psi(\sigma t-\tau).} \tag{23}$$

This two-parameter family of functions, labeled by the scale factor and the time shift, is called the *wavelet family* generated by $\psi(t)$.

Note: In most of the wavelet literature, including [5], it is customary to define the wavelets by

$$\psi_{s,\tau'}(t) = s^{-1/2}\psi\left(\frac{t-\tau'}{s}\right). \tag{24}$$

These are related to (23) by setting $s=1/\sigma$ and $\tau'=\tau/\sigma$. The factor $s^{-1/2}$ in (24) is actually unnecessary, since its only purpose is to ensure that all the wavelets have the same energy, i.e., $\|\psi_{s,\tau'}\| = \|\psi\|$ for all values of (s,τ'). The entire theory works equally well without it. (Its absence is made up for later, in the reconstruction formula; see [5], pages 63 and 68.) With our normalization, a factor of $\sqrt{\sigma}$ will show up instead in Equation (26) below. We use the form (23) because it is simpler and more directly related to the radar problem. While the parameter s in (24) can be interpreted as the *time scale* (small s means fine scale and large s means coarse scale), our parameter σ can be interpreted as the *time resolution:* Large σ means high resolution, and small σ means low resolution.

With (22), the ambiguity functional reduces to

$$\tilde{\chi}(\sigma,\tau) = A\langle\,\psi_{\sigma,\tau}\,,\chi\,\rangle = A\int_{-\infty}^{\infty} dt\;\psi(\sigma t-\tau)\,\chi(t), \tag{25}$$

which is (apart from the factor A) the *wavelet transform* of the return χ with respect to the wavelet family $\psi_{\sigma,\tau}$. This is known as the *wideband ambiguity function;* see [5,20–26]. We have thereby established a connection between the ambiguity functional $\tilde{\chi}[\beta]$ and some of the recent radar literature.

The wideband ambiguity function $\tilde{\chi}(\sigma,\tau)$ is used to estimate the range r_0 and range rate v (which characterize our *restricted* trajectory $\beta(t)$ in (18)) in exactly the same way as the general ambiguity functional $\tilde{\chi}[\beta]$ was used to estimate a *general* trajectory $\beta(t)$. Again Schwarz's inequality implies that

$$|\tilde{\chi}(\sigma,\tau)| \leq A\,\|\psi_{\sigma,\tau}\|\,\|\chi\|,$$

and equality holds if and only if $\chi(t) = k\,\psi_{\sigma,\tau}(t)$ for some constant k. But

$$\|\psi_{\sigma,\tau}\|^2 = \int_{-\infty}^{\infty} dt\ \psi(\sigma t - \tau)^2 = \sigma^{-1}\|\psi\|^2.$$

Therefore, to find the correct values of (σ,τ), we can maximize $\sqrt{\sigma}\,|\tilde{\chi}(\sigma,\tau)|/A$ (noting that A depends on σ through v). Equivalently, we can minimize the *wideband error function,* defined by

$$\mathcal{E}(\sigma,\tau) = 1 - \frac{|\langle\,\psi_{\sigma,\tau}\,,\chi\,\rangle|}{\|\psi_{\sigma,\tau}\|\,\|\chi\|} = 1 - \frac{\sqrt{\sigma}\ |\tilde{\chi}(\sigma,\tau)|}{A\,\|\psi\|\,\|\chi\|}\ , \tag{26}$$

which satisfies

$$0 \leq \mathcal{E}(\sigma,\tau) \leq 1 \quad \text{and} \quad \mathcal{E}(\sigma,\tau) = 0 \Longleftrightarrow \chi(t) = k\,\psi_{\sigma,\tau}(t).$$

$\mathcal{E}(\sigma,\tau)$ estimates the error in the wideband parameters (σ,τ) which characterize the trajectory (18). Once a good estimate is found, the range and range rate are determined by solving (20), which gives

$$r_0 = \frac{c\,\tau}{1+\sigma} \quad \text{and} \quad v = \left(\frac{1-\sigma}{1+\sigma}\right)c\,. \tag{27}$$

We have seen above that a uniform motion of the target results in a *scaling* of the return. But the common treatment of the Doppler effect in most of the literature is as a *uniform frequency shift.* How are these two views related? To see this, suppose for a moment that $\psi(t)$ is time-harmonic, i.e., $\psi(t) = e^{2\pi\,ift}$ for some frequency f. Then

$$\psi_{\sigma,\tau}(t) = e^{2\pi\,if(\sigma t-\tau)} = e^{-2\pi\,if\tau}\ e^{2\pi\,i\sigma ft}\ . \tag{28}$$

Thus $\psi_{\sigma,\tau}(t)$ is also time-harmonic, but with frequency σf. Since an arbitrary signal $\psi(t)$ can be synthesized from such time-harmonic signals, it follows that *every frequency component in $\psi(t)$ is scaled by $f \to \sigma f$.* If $0 < v < c$, then $0 < \sigma < 1$ and all frequencies are scaled *down,* as expected. If $-c < v < 0$, then $\sigma > 1$ and all frequencies are scaled *up.* But note that "down" and "up" mean *toward and away from zero, respectively, and not to the left or right.* There are two separate ways in which the frequency scaling differs from a frequency shift: (a) Negative frequencies move in the *opposite* direction from positive frequencies, and (b) high frequencies move more than low frequencies. The first difficulty is easily overcome by noting that since $\psi(t)$ was assumed to be real, its Fourier transform

$$\hat{\psi}(f) = \int_{-\infty}^{\infty} dt\ \ e^{-2\pi\,ift}\,\psi(t)$$

satisfies the "reality condition"

$$\hat{\psi}(-f) = \hat{\psi}(f)^*, \tag{29}$$

where the asterisk denotes complex conjugation. Therefore we can ignore the negative-frequency components in $\psi(t)$ without any loss of information. That is, replace

$$\psi(t) = \int_{-\infty}^{\infty} df \;\; e^{2\pi\, i f t}\, \hat{\psi}(f) \tag{30}$$

by

$$\psi^+(t) = \int_{0}^{\infty} df \;\; e^{2\pi\, i f t}\, \hat{\psi}(f). \tag{31}$$

Note that $\psi^+(t)$ is necessarily *complex* since its Fourier transform $\hat{\psi}^+(f)$ vanishes for $f < 0$, and so it cannot satisfy a reality condition like (29) unless it vanishes identically. It follows from (29) that the original signal can be recovered by taking the real part:

$$\psi(t) = \psi^+(t) + \psi^+(t)^* = 2\,\mathrm{Re}\,\psi^+(t),$$

confirming that no information has been lost. (Similarly, $2\,\mathrm{Im}\,\psi^+(t)$ is the *Hilbert transform* of $\psi(t)$ [27].) Furthermore, if in the integral (31) representing $\psi^+(t)$ we formally replace the real time parameter t by $t + iu$ with $u > 0$, then the integrand gains an extra factor $e^{-2\pi\, f u}$ which decays exponentially for $f > 0$. Consequently, the integral defines an *analytic function* $\psi^+(t+iu)$ *in the upper-half complex time plane.* For this reason Dennis Gabor, who introduced the positive-frequency parts of real functions into signal analysis in 1946, called $\psi^+(t)$ the *analytic signal* of $\psi(t)$ [28].* With $\psi(t)$ replaced by its analytic signal $\psi^+(t)$, all the frequency components move in the same direction under scaling: To the left when $0 < \sigma < 1$, and to the right when $\sigma > 1$. It only remains to deal with difficulty (b) above, i.e., the fact that high frequencies shift more than low frequencies. Clearly, the only way to overcome this is to assume that $\psi(t)$ contains only a narrow band of frequencies. This will be the first part of our final assumption.

D. Narrowband approximation:
1. *The spectrum of* $\psi^+(t)$ *is concentrated in a narrow frequency band centered around a high frequency* f_c. That is,

$$\hat{\psi}^+(f) = 0 \quad \text{unless} \quad |f - f_c| \le b\,, \quad \text{where} \quad b \ll f_c\,. \tag{32}$$

(Thus $\hat{\psi}(f) = 0$ outside the *two* frequency bands $|f - f_c| \le b$ and $|f + f_c| \le b$.) A function $\psi(t)$ satisfying (32) will be called a *narrowband signal.* It can be interpreted as a slowly

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* This idea is generalized to n dimensions by the *analytic-signal transform.* For example, functions in space-time such as acoustic and electromagnetic waves can be extended to complex space-time; see [5,29].

varying "message" (with *bandwidth* $2b$) coded onto a rapidly oscillating carrier with *carrier frequency* f_c, like an audio signal imprinted on a radio wave. Then

$$\psi^+(t) = \int_{f_c-b}^{f_c+b} df\ e^{2\pi\, ift}\,\hat{\psi}(f) = e^{2\pi\, if_c t} \int_{-b}^{b} df\ e^{2\pi\, ift}\,\hat{\psi}(f+f_c). \tag{33}$$

The function

$$\Psi(t) = \int_{-b}^{b} df\ e^{2\pi\, ift}\,\hat{\psi}(f+f_c) = e^{-2\pi\, if_c t}\,\psi^+(t) \tag{34}$$

contains only low frequencies ($|f| \le b$), hence it can be *sampled* much more slowly than either $\psi(t)$ or its analytic signal $\psi^+(t)$. In fact, the Nyquist sampling rate for $\psi(t)$ is $2(f_c+b)$, while that for $\Psi(t)$ is $2b \ll 2f_c$. Equation (34) states that $\Psi(t)$ is a *demodulated* form of $\psi^+(t)$. The carrier has been removed, so that only the "message" remains. $\Psi(t)$ is called the *video signal* of $\psi(t)$. It is *complex* in general, and its real and imaginary parts are called the *in-phase (I) component* and the *quadrature (Q) component,* respectively:

$$\begin{aligned} I(t) &= \operatorname{Re}\Psi(t) = \text{ In-phase component} \\ Q(t) &= \operatorname{Im}\Psi(t) = \text{ Quadrature component .} \end{aligned} \tag{35}$$

In practice, $\psi(t)$ varies much too rapidly (at radio frequency) for sampling, and something very similar to the transformation $\psi(t) \to \psi^+(t) \to \Psi(t)$ must be performed at the hardware level before any digital processing can be applied or the signal can be used to control a video display. For a clear explanation of this, with wonderful illustrations, I recommend Stimson [30].

Let us now see how the video signal is affected by the reflection. Since reflections are now represented by $\psi(t) \to A\psi_{\sigma,\tau}(t)$, we have to compute the video signal of $\psi_{\sigma,\tau}(t)$. To do this, we must first find the analytic signal $\psi^+_{\sigma,\tau}(t)$ of $\psi_{\sigma,\tau}(t)$. But it is easily seen that the operation $\psi(t) \to \psi^+(t)$ (which projects to the positive-frequency spectrum by filtering out the negative frequencies) *commutes* with time shifts and time scaling (for *positive* scale factors). Therefore, *the analytic signal of $\psi_{\sigma,\tau}(t)$ is a shifted and scaled version of $\psi^+(t)$:*

$$\psi^+_{\sigma,\tau}(t) = \psi^+(\sigma t - \tau). \tag{36}$$

By (33) and (34),

$$\psi^+_{\sigma,\tau}(t) = e^{2\pi\, if_c(\sigma t-\tau)}\,\Psi(\sigma t - \tau). \tag{37}$$

Now

$$f_c\,\sigma = f_c\left(\frac{c-v}{c+v}\right) = f_c - \frac{2f_c v}{c+v} \equiv f_c + \phi,$$

where

$$\phi = \phi(v) = -\frac{2f_c v}{c+v} \tag{38}$$

is the *Doppler frequency shift* suffered by the carrier. Thus

$$\begin{aligned}\psi^+_{\sigma,\tau}(t) &= e^{2\pi\, i(f_c+\phi)t}\; e^{-2\pi\, if_c\tau}\, \Psi(\sigma t-\tau)\\ &= e^{2\pi\, if_c(t-\tau)}\; e^{2\pi\, i\phi t}\, \Psi(\sigma t-\tau).\end{aligned}\tag{39}$$

To complete the narrowband approximation, we would like to replace $\Psi(\sigma t-\tau)$ by $\Psi(t-\tau)$ in (39), since the Doppler effect is then characterized by the frequency shift ϕ alone. Note that by (34),

$$\Psi(\sigma t-\tau)=\int_{-b}^{b} df\;\; e^{2\pi\, if(\sigma t-\tau)}\, \hat{\psi}(f+f_c).\tag{40}$$

The replacement $\Psi(\sigma t-\tau)\to\Psi(t-\tau)$ amounts to $f\sigma\to f$ in the exponent of (40). But

$$f\sigma-f=f\left(\frac{c-v}{c+v}-1\right)=-\frac{2fv}{c+v}\ .$$

Therefore, to justify the substitution $f\sigma\to f$, we need a second narrowband assumption.

2. *The speed of the target is much smaller than the propagation speed of the waves:*

$$|v|\ll c\,,\qquad \text{i.e.,}\qquad \sigma\equiv\frac{c-v}{c+v}\approx 1.\tag{41}$$

If (41) holds, then for $|f|\le b$ we have

$$|f\sigma-f|\le\frac{2b|v|}{c+v}\approx\frac{2b|v|}{c}\ll 2b.$$

That is, the error in the phase of the integrand (40) incurred by the substitution $f\sigma\to f$ is small compared with the bandwidth $2b$, and

$$\Psi(\sigma t-\tau)\approx\Psi(t-\tau),\tag{42}$$

if t is not too large. (We assume that $\Psi(t)$ is effectively time-limited, so that (42) needs to be proved only for a bounded time interval, say $|t|\le T$. A more careful analysis must be performed in order to see exactly in what sense (42) holds. For example, note that we *cannot* similarly replace $f_c(\sigma t-\tau)$ by $f_c(t-\tau)$ in the phase factor in (37), since $f_c\,\sigma=f_c+\phi$ and ϕ need not be small compared to $2b$ because if contains f_c as a factor! See Swick [21].) Thus (39) gives

$$\psi^+_{\sigma,\tau}(t)\to\; e^{2\pi\, if_c t}\,\Psi_{\phi,\tau}(t),\tag{43}$$

where

$$\Psi_{\phi,\tau}(t)=\; e^{-2\pi\, if_c\tau}\;\, e^{2\pi\, i\phi t}\,\Psi(t-\tau)\tag{44}$$

is the video signal of $\psi_{\sigma,\tau}(t)$, which is seen to be a *delayed and modulated* version of the video signal $\Psi(t)$ of $\psi(t)$.

In the narrowband approximation, the reflection $\psi(t) \rightarrow A\psi_{\sigma,\tau}(t)$ is represented by the transformation $\Psi(t) \rightarrow A\Psi_{\phi,\tau}(t)$ of the video signal, where the Doppler shift ϕ has replaced the scale factor σ.

The family $\psi_{\sigma,\tau}(t)$ of wavelets, labeled by the *wideband parameters* (σ, τ), has thus been replaced by the family $\Psi_{\phi,\tau}(t)$ of video signals, labeled by the *narrowband parameters* (ϕ, τ). Note that in view of (41) the attenuation factor A, the time shift (20), and the Doppler shift (38) can be approximated by

$$A \approx \frac{g_\gamma g_\alpha}{16\pi^2 r_0^2}\ , \quad \tau \approx \frac{2r_0}{c}\ , \quad \text{and} \quad \phi \approx -\frac{2f_c\, v}{c} = -\frac{2v}{\lambda_c}\ , \tag{45}$$

where $\lambda_c = c/f_c$ is the carrier wavelength.

All the ingredients are now in place. If the return has the form $A\psi_{\sigma,\tau}(t)$ and the narrowband assumptions *D1–D2* are valid, then the video signal of the return has the form $A\Psi_{\phi,\tau}(t)$ for some unknown Doppler frequency ϕ and time delay τ. If we can find ϕ and τ, then (45) gives the range and range rate:

$$r_0 \approx \frac{1}{2}\, c\tau \quad \text{and} \quad v \approx -\frac{1}{2}\, \lambda_c\, \phi\, . \tag{46}$$

The measured return $\chi(t)$ is related to its video signal $X(t)$ by

$$\chi(t) = e^{2\pi\, i f_c t}\, X(t). \tag{47}$$

It is easy to see (using Parseval's relation and the reality condition (29)) that the wideband ambiguity function (25) can be expressed directly in terms of the analytic signals $\psi^+_{\sigma,\tau}(t)$ and $\chi^+(t)$ by

$$\begin{aligned} \tilde{\chi}(\sigma, \tau) &\equiv A\langle\, \psi_{\sigma,\tau}\, , \chi\, \rangle = 2A\, \mathrm{Re}\, \langle\, \psi^+_{\sigma,\tau}\, , \chi^+\, \rangle \\ &= 2A\, \mathrm{Re} \int_{-\infty}^{\infty} dt\ \psi^+_{\sigma,\tau}(t)^*\, \chi^+(t), \end{aligned} \tag{48}$$

where the complex conjugation is necessary because the analytic signals are complex. Inserting (43) and (47) into (48), we have in the narrowband approximation

$$\tilde{\chi}(\sigma, \tau) \rightarrow 2A\, \mathrm{Re}\, \langle\, \Psi_{\phi,\tau}\, , X\, \rangle = 2A\, \mathrm{Re}\, \tilde{X}(\phi, \tau), \tag{49}$$

where

$$\tilde{X}(\phi, \tau) = \langle\, \Psi_{\phi,\tau}\, , X\, \rangle = e^{2\pi\, i f_c \tau} \int_{-\infty}^{\infty} dt\ \ e^{-2\pi\, i\phi t}\, \Psi(t - \tau)^*\, X(t) \tag{50}$$

is the *narrowband ambiguity function.* It matches the video signal $X(t)$ of the return with the video signal $\Psi_{\phi,\tau}(t)$ of the wavelet $\psi_{\sigma,\tau}(t)$. By Schwarz's inequality,

$$|\tilde{X}(\phi, \tau)| \le \|\Psi_{\phi,\tau}\|\, \|X\| = \|\Psi\|\, \|X\|, \tag{51}$$

where $\|\Psi_{\phi,\tau}\|^2$ and $\|X\|^2$ are the energies of $\Psi_{\phi,\tau}(t)$ and $X(t)$, and $\|\Psi_{\phi,\tau}\| = \|\Psi\|$ follows immediately from (44). Thus, to estimate (ϕ, τ) we can maximize $|\tilde{X}(\phi, \tau)|$. Or we can define the *narrowband error function*

$$\mathcal{E}(\phi, \tau) = 1 - \frac{|\tilde{X}(\phi, \tau)|}{\|\Psi\| \, \|X\|} \, , \tag{52}$$

which satisfies

$$0 \leq \mathcal{E}(\phi, \tau) \leq 1 \quad \text{and} \quad \mathcal{E}(\phi, \tau) = 0 \Longleftrightarrow X(t) = k \, \Psi_{\phi,\tau}(t)$$

for some (possibly complex) constant k. $\mathcal{E}(\phi, \tau)$ represents the error in the narrowband parameters, which is to be minimized in order to estimate (ϕ, τ).

In (35) we have defined the in-phase and quadrature components of a video signal. We are now in a position to see their meaning. According to (44), the video signal $\Psi_{\phi,\tau}(t)$ of the return *rotates* through the in-phase and quadrature components of $\Psi(t - \tau)$ at the Doppler frequency ϕ. If we sample *only* the in-phase or the quadrature component of the video signal of the return, this will give us $|\phi|$ but not the *sign* of ϕ. We can then compute the *speed* $|v|$ of the target but not its *direction* (toward or away from the radar site). Obviously this is a very important piece of information! Thus, *both* components of the video signal must be sampled in practice, i.e., *we must necessarily deal with complex signals!* (A similar conclusion applies to the nonrelativistic approximation of quantum mechanics, which has much in common with the narrowband approximation; see [29], Section 4.6.)

When is the narrowband approximation appropriate? In *radar,* where c is the speed of light, it can always be assumed that $|v| \ll c$. It is also usually true that $\psi(t)$ is narrowband, although with modern technology ultrawideband radar signals are also possible. This is a topic of current research. (A review is given in [31]; see also [32].) But in *sonar,* where c is the speed of sound (usually in sea water), *neither* of the assumptions *D1–2* are valid in general. Sonar signals are typically wideband, and although targets usually move much slower than sound, situations may occur where v/c is not small. For example, when a ship and a torpedo are closing in at maximum speeds, the relative velocity may be a significant fraction of c.*

This completes our journey from the ambiguity functional to the wideband ambiguity function and its narrowband version. As a byproduct, we have the following remarkable observation. Equation (50) shows that, apart from the constant phase factor $e^{2\pi \, i f_c \tau}$ (which can be ignored since it does not contribute to the error function $\mathcal{E}(\phi, \tau)$), the narrowband ambiguity function $\tilde{X}(\phi, \tau)$ is the *windowed Fourier transform* of the video signal $X(t)$ of the return. Therefore, ambiguity functions provide a natural link between time-scale analysis and time-frequency analysis.

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* I thank Ralph Hippenstiel of the Naval Postgraduate School in Monterey, CA for this observation.

In the wideband approximation (Assumptions A–C), the ambiguity functional reduces to a wavelet (time-scale) transform of the return $\chi(t)$. In the narrowband approximation (Assumptions *D1*–*D2*), it further reduces to a windowed Fourier (time-frequency) transform of the video signal $X(t)$ of the return.

4. Conclusions

Physical wavelets have been shown to generalize ordinary ("mathematical") wavelets, such as those given in (23) and (24). The physical wavelets originate in real-world problems, and they reduce to ordinary wavelets when their sources and targets undergo linear motions. The idea of ambiguity functions has been generalized to the context of such wavelets. The resulting scheme can be used to analyze general (accelerating or nonlinear) target motions, as well as general and independent motions by the transmitting and receiving antennas. However, the transmitters, targets, and receivers considered here have all been assumed to be *point objects.* The next objective of this research is to examine physical wavelets with *extended* sources rather than point sources. Such wavelets can have some *directivity,* hence they provide more realistic models for actual antenna or array outputs. They can be used to probe extended objects rather than merely point objects, and it is hoped that this will provide a method not only for *tracking* but also for *imaging.*

5. Appendix: Derivation of Equation (19)

To prove (19), recall that

$$\alpha(t) = \gamma(t) = (\mathbf{0}, t) \quad \text{and} \quad \beta(t) = ((r_0 + vt)\mathbf{n}, t) = (r(t)\mathbf{n}, t).$$

To simplify the notation we also define

$$\rho(t) = \frac{r(t)}{c} = \frac{r_0 + vt}{c}\ .$$

Denoting the wavelet received at $\beta(t)$ (before re-emission) by $\psi'(t)$, we have

$$\begin{aligned} \psi'(t) \equiv (R_\beta E_\alpha \psi)(t) &= g_\beta \int dt'\, G(\beta(t) - \alpha(t'))\, \psi(t') \\ &= g_\beta \int dt'\, G(r(t)\mathbf{n}, t - t')\, \psi(t') \\ &= g_\beta \int dt'\ \frac{\delta(t - t' - \rho(t))}{4\pi r(t)}\ \psi(t') \\ &= g_\beta \frac{\psi(t - \rho(t))}{4\pi r(t)}\ . \end{aligned} \tag{53}$$

Therefore the reflected wavelet observed back at $\gamma(t) = \alpha(t)$ is

$$\begin{aligned}\psi_\beta(t) &= g_\gamma g_\beta \int dt'\, G(\alpha(t) - \beta(t'))\, \psi'(t') \\ &= g_\gamma g_\beta \int dt'\, G(-r(t')\mathbf{n}, t - t')\, \psi'(t') \\ &= g_\gamma g_\beta \int dt'\; \frac{\delta(t - t' - \rho(t'))}{4\pi r(t')} \cdot \frac{\psi(t' - \rho(t'))}{4\pi r(t')}\ .\end{aligned} \tag{54}$$

But

$$\begin{aligned}t - t' - \rho(t') &= \frac{c(t-t') - r_0 - vt'}{c} \;= \frac{ct - r_0 - (c+v)t'}{c} \\ &= -\frac{c+v}{c}\left[t' - \frac{c\,t - r_0}{c+v}\right].\end{aligned} \tag{55}$$

Using the shorthand

$$T = \frac{c\,t - r_0}{c+v}\ ,$$

(55) implies that

$$\delta(t - t' - \rho(t')) = \frac{c}{c+v}\ \delta(t' - T).$$

Therefore (54) gives

$$\psi_\beta(t) = g_\gamma g_\beta \frac{c}{c+v} \cdot \frac{\psi\left(T - \rho\left(T\right)\right)}{\left[4\pi r\left(T\right)\right]^2}\ . \tag{56}$$

But

$$r(T) = r_0 + v\left(\frac{c\,t - r_0}{c+v}\right) = \frac{c}{c+v}\ r(t),$$

hence

$$\begin{aligned}T - \rho(T) &= T - \frac{r(T)}{c} \;= T - \frac{r(t)}{c+v} \;= \frac{c\,t - r_0}{c+v} - \frac{r_0 + vt}{c+v} \\ &= \left(\frac{c-v}{c+v}\right) t - \frac{2r_0}{c+v}\ .\end{aligned}$$

Thus (56) becomes

$$\psi_\beta(t) = \frac{g_\gamma g_\beta (1 + v/c)}{16\pi^2 r(t)^2}\ \psi\left(\left(\frac{c-v}{c+v}\right) t - \frac{2r_0}{c+v}\right),$$

which is indeed (19).

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