

## SHORT-PULSE RADAR VIA ELECTROMAGNETIC WAVELETS \*

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### INTRODUCTION

The new theory of physical wavelets makes it possible to perform radar and sonar analysis directly in the space-time domain, based on fundamental principles underlying the emission, reflection, and reception of electromagnetic and acoustic waves<sup>1,2</sup>. Being independent of the Fourier transform and even of the usual (affine) wavelet transform, this formalism is therefore equally well suited for ultrawideband or short-pulse radar as for narrowband or continuous-wave radar. However, Fourier analysis does have a natural place in this theory and can be used easily when spectral questions are of interest.

A transmitting antenna following an arbitrary (possibly accelerating or nonlinear) space-time trajectory  $\alpha(t)$  emits a physical (acoustic or electromagnetic) wavelet which is propagated in space by the appropriate Green function. This defines an *emission operator*  $E_\alpha$  which, acting on any time signal  $\psi(t)$ , gives the emitted space-time wave  $(E_\alpha\psi)(\mathbf{r}, t)$ . The *reception operator*  $R_\alpha$  is dual to  $E_\alpha$ , measuring any incident space-time wave  $F(\mathbf{r}, t)$  along the given antenna trajectory  $\alpha(t)$  to produce the received time signal  $(R_\alpha F)(t)$ . Reflection is modeled as reception followed by re-emission, i.e., by the operator  $E_\alpha R_\alpha$  transforming any incident space-time wave to the reflected space-time wave. Let the receiving antenna follow another arbitrary space-time trajectory  $\gamma(t)$  (possibly different from the trajectory  $\alpha(t)$  of the transmitter), let the target follow a third arbitrary space-time trajectory  $\beta_T(t)$ , and let the transmitted and received signals be  $\psi(t)$  and  $\chi(t)$ , respectively. The objective is to estimate the target trajectory  $\beta_T(t)$  from a knowledge of  $\alpha(t)$ ,  $\gamma(t)$ ,  $\psi(t)$  and  $\chi(t)$ . This is achieved by maximizing the modulus of the the *normalized ambiguity functional*  $\tilde{\chi}_N(\beta)$ , obtained by matching the *actual* return  $\chi(t)$  with the *computed* return due to a *trial trajectory*  $\beta(t)$ . When the radar is monostatic and the target is assumed to move uniformly in the radial direction, then  $\tilde{\chi}_N(\beta)$  reduces to the usual *wideband ambiguity function*, which is just the ordinary *time-scale (wavelet) transform* of  $\chi(t)$  with  $\psi(t)$

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as the basic wavelet. In the narrowband approximation, it reduces further to the usual time-frequency ambiguity function, which is a windowed Fourier transform of the *video signal* of the return. This shows that our “physical wavelet analysis” is a generalization of the usual (“mathematical”) wavelet analysis, which is in turn a generalization of time-frequency analysis. In particular, our analysis applies equally well to *bistatic wideband radar*, where the Doppler effect can no longer be represented simply by scaling and hence the usual affine wavelet analysis breaks down.

In Reference 2, the transmitting and receiving antennas and the target were all assumed to be *points*, so that the trajectories  $\alpha(t)$ ,  $\beta(t)$ , and  $\gamma(t)$  fully describe their motions. Consequently, the emitted and reflected waves are omnidirectional and hence are not very useful in practice. In this paper we generalize the above model to include *extended* antennas and targets, following an idea introduced by Heyman and Felsen<sup>3</sup>. The associated radar beams are much more useful since they have a measure of directivity.

## EXTENDED PHYSICAL WAVELETS AND AMBIGUITY FUNCTIONALS

Suppose we are given an antenna located at the space point  $\mathbf{x}$ , emitting the response to an impulse at time  $t$ . Ignoring polarization for simplicity, the resulting wave at the observation point  $\mathbf{x}'$  at time  $t'$  can be represented by a solution of the scalar wave equation, which we write as

$$K(\mathbf{x}', t' | \mathbf{x}, t),$$

with a source distribution appropriate to the antenna. If the antenna is very small (essentially a point) and omnidirectional, then  $K(\mathbf{x}', t' | \mathbf{x}, t)$  is well approximated by the retarded Green function

$$G(\mathbf{x}' - \mathbf{x}, t' - t) = \frac{\delta(t' - t - |\mathbf{x}' - \mathbf{x}|/c)}{4\pi|\mathbf{x}' - \mathbf{x}|},$$

where  $c$  is the speed of light. Following Heyman and Felsen<sup>3</sup>, a simple model can be formulated for an *extended* antenna by allowing *complex* antenna space-time coordinates  $\mathbf{x} \rightarrow \mathbf{z} = \mathbf{x} + i\mathbf{y}$  and  $t \rightarrow u = t + is$  and taking  $K(\mathbf{x}', t' | \mathbf{z}, u)$  to be an analytic extension of the above retarded Green function.\* Heyman and Felsen showed that for  $|\mathbf{y}| < cs$ ,  $K(\mathbf{x}', t' | \mathbf{z}, u)$  can be interpreted as a *pulsed beam field* emitted by a circular disk of radius  $|\mathbf{y}|$  in the direction of  $\mathbf{y}$ . Thus  $\mathbf{y}$  is a convenient “handle” by which the radius and orientation of the antenna can be controlled, without having to construct a messy model for the antenna involving a continuous distribution of point sources. More complicated extended antennas and arrays can be modeled by a distribution (continuous or discrete) of such complex source points.

To keep the notation uncluttered, we combine the space and time coordinates into a single symbol:

$$x' \equiv (\mathbf{x}', t') \in \mathbf{R}^4, \quad z \equiv (\mathbf{z}, u) = x + iy \in \mathbf{C}^4,$$

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\* Note that our convention differs from that of Reference 3 in that our positive-frequency time-harmonic waves vary as  $e^{i\omega t}$  rather than  $e^{-i\omega t}$ . For this reason, the analytic continuation of the retarded Green function (using the analytic signal of the delta function) is to the upper-half time plane ( $s > 0$ ) rather than the lower-half time plane. This is consistent with the convention used in Reference 1.

where

$$x = (\mathbf{x}, t) \quad \text{and} \quad y = (\mathbf{y}, s).$$

The condition  $|\mathbf{y}| < cs$  means that the imaginary space-time four-vector  $y$  belongs to the *future cone*, so that  $z$  actually belongs to the complex *future tube*<sup>1,4</sup>

$$z \in \mathcal{T}_+ \equiv \{x + iy \in \mathbf{C}^4 : x \in \mathbf{R}^4 \text{ and } y = (\mathbf{y}, s) \text{ with } |\mathbf{y}| < cs\},$$

which is a four-dimensional generalization of the upper-half complex time plane.

Suppose now that our antenna executes an arbitrary motion, including possible rotations and accelerations. Using the complex source coordinates, this can be parameterized as

$$z = \alpha(t) = x(t) + iy(t), \quad \text{where } x(t) = (\mathbf{x}(t), t) \in \mathbf{R}^4 \text{ and } y(t) = (\mathbf{y}(t), s). \quad (1)$$

It is reasonable (but mathematically unnecessary) to assume that the radius of the antenna remains constant during the motion, so that  $|\mathbf{y}(t)| = |\mathbf{y}(0)| \equiv R < cs$ , although the direction of  $\mathbf{y}(t)$  may vary to allow tracking, scanning, etc. While executing this motion, the antenna is fed an input time signal  $\psi(t)$ . Then the output beam is

$$\Psi_\alpha(x') = \int_{-\infty}^{\infty} dt K(x' | \alpha(t)) \psi(t).$$

For reasons explained in Reference 2, we call  $\Psi_\alpha(x')$  the *extended physical wavelet* generated by  $\psi(t)$  along the antenna motion  $\alpha(t)$ . Given  $\alpha(t)$ , we define the *emission operator*  $E_\alpha$  as the operator transforming the time signal  $\psi(t)$  to the space-time wave  $\Psi_\alpha(x')$ , i.e.,

$$(E_\alpha \psi)(x') \equiv \int_{-\infty}^{\infty} dt K(x' | \alpha(t)) \psi(t). \quad (2)$$

Thus  $E_\alpha$  takes a function of one variable (the input signal) to a function of four variables (the output beam). On the other hand, if the antenna is used as a receiver, it converts space-time waves into time signals. Again, assume that the complex antenna motion  $\alpha(t)$  is given as in (1). Then the simplest model for the received signal due to an incident wave  $F(x')$  is

$$(R_\alpha F)(t) = g_\alpha F(\alpha(t)), \quad (3)$$

where  $g_\alpha$  is a “gain factor.” Thus  $R_\alpha$  simply measures the field along the complex trajectory  $\alpha(t)$ . More complicated receivers can be formulated which measure derivatives of  $F$  along  $\alpha(t)$ . (In the full electromagnetic formalism, for example,  $R_\alpha$  could measure the induced current rather than the field.) Since  $\alpha(t)$  is complex, the “evaluation” of the field  $F(x')$  at  $x' = \alpha(t)$  must be defined in (3). For this we use the *analytic-signal transform* of  $F$ , which extends  $F$  to complex space-time<sup>1,4</sup>:

$$F(x + iy) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} F(x + \tau y).$$

When  $y = (\mathbf{0}, s)$  with  $s > 0$ ,  $F(x + iy)$  reduces to the usual Gabor analytic signal  $F(\mathbf{x}, t + is)$  corresponding to  $F(\mathbf{x}, t)$ , with  $\mathbf{x}$  regarded as an external parameter; this function is analytic in the upper-half complex time plane. It is further shown in Reference 1 that if  $F(x')$  is any

solution of the homogeneous wave equation (or Klein-Gordon equation<sup>4</sup>), then  $F(x + iy)$  is analytic in the future tube  $\mathcal{T}_+$  (i.e.,  $|\mathbf{y}| < cs$ ). The reception operator then evaluates the analytic-signal transform  $F(x + iy)$  in its region of analyticity.

With emission and reception modeled by (2) and (3), we are almost ready to formulate a general radar problem. The only missing element is a model for *reflection*. In the spirit of regarding a scattered electromagnetic wave as being emitted by the current induced on the scatterer by the incident wave, we propose the following model: Suppose we are given an oriented circular “target” disk executing a motion described by a complex space-time trajectory  $\alpha(t) = x(t) + iy(t)$  as in (1). Again, we interpret the imaginary position vector  $\mathbf{y}(t)$  as defining the radius and orientation of the disk. (To say that the disk is “oriented” means that its two sides are not equivalent; for example, one side could be reflective while the other side is not. Then every unit vector  $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$  corresponds to a unique orientation of the disk. This is useful if, for example, we approximate a complicated target by patching together disks of various sizes and orientations, as in Section 10.2 of Reference 1; their non-reflecting sides should then be oriented towards the interior.) A given space-time wave  $F(x')$  will now be assumed to be reflected from the disk as follows: First the disk acts as a *receiver*, then as a *transmitter*. Thus the reflected wave is

$$F_{\text{refl}}(x') = (E_\alpha R_\alpha F)(x') = g_\alpha \int_{-\infty}^{\infty} dt K(x' | \alpha(t)) F(\alpha(t)).$$

Note that in the present context, the original “gain factor”  $g_\alpha$  is re-interpreted as a *reflection coefficient*. When a complicated target is patched together from circular targets of various radii and orientations, the reflection coefficient becomes a function defined over the target surface as desired.

The ambiguity functional formalism developed in Reference 2 generalizes easily and naturally to the present setting of extended physical wavelets. Given the outgoing time signal  $\psi$  and the motions  $\alpha, \beta$ , and  $\gamma$  of the transmitter, target, and receiver (all complex), our model for the time signal received at  $\gamma$  is

$$\begin{aligned} \psi_\beta(t'') &= (R_\gamma E_\beta R_\beta E_\alpha \psi)(t'') \\ &= g_\gamma g_\beta \iint dt' dt K(\gamma(t'') | \beta(t')) K(\beta(t') | \alpha(t)) \psi(t). \end{aligned} \quad (4)$$

Of course, the received signal depends functionally on all three trajectories  $\alpha, \beta, \gamma$ , as is evident from the right-hand side of (4). But to simplify the notation, we have suppressed the dependence on the *known* trajectories  $\alpha$  and  $\gamma$  and displayed only the dependence on the target trajectory  $\beta$ . To estimate the *actual* target trajectory  $\beta_T(t)$ , we compute  $\psi_\beta(t)$  for a *trial trajectory*  $\beta(t)$  and match the result with the actual return  $\chi(t)$  by taking the inner product of the two time signals. We denote the result by  $\tilde{\chi}(\beta)$ , which we call the *ambiguity functional* of the return:

$$\begin{aligned} \tilde{\chi}(\beta) &\equiv \langle \chi, \psi_\beta \rangle \equiv \int_{-\infty}^{\infty} dt'' \chi(t'') \psi_\beta(t'') \\ &= \iiint dt'' dt' dt \chi(t'') K(\gamma(t'') | \beta(t')) K(\beta(t') | \alpha(t)) \psi(t). \end{aligned} \quad (5)$$

(We assume that  $\psi(t)$  and  $\chi(t)$  are real; if they are complex, then  $\chi(t)$  should be replaced by its complex conjugate in (5).) Assuming that  $\psi_\beta(t)$  and  $\chi(t)$  have finite energies  $\|\psi_\beta\|^2$

and  $\|\chi\|^2$ , the Schwarz inequality implies that

$$\begin{aligned} |\tilde{\chi}(\beta)| &= |\langle \chi, \psi_\beta \rangle| \leq \|\chi\| \|\psi_\beta\| \\ |\tilde{\chi}(\beta)| &= \|\chi\| \|\psi_\beta\| \iff \chi(t) = C\psi_\beta(t). \end{aligned} \quad (6)$$

Therefore, to estimate the true target trajectory  $\beta_T(t)$ , we need to maximize the *normalized ambiguity functional*

$$\tilde{\chi}_N(\beta) \equiv \frac{\tilde{\chi}(\beta)}{\|\psi_\beta\|}.$$

By (6),

$$|\tilde{\chi}_N(\beta)| \leq \|\chi\| \quad \text{and} \quad |\tilde{\chi}_N(\beta)| = \|\chi\| \iff \chi(t) = C\psi_\beta(t).$$

Equivalently, we can minimize the *error functional* defined by

$$\mathcal{E}(\beta) \equiv 1 - \frac{|\chi(\beta)|}{\|\chi\| \|\psi_\beta\|},$$

since the Schwarz inequality states that

$$0 \leq \mathcal{E}(\beta) \leq 1 \quad \text{and} \quad \mathcal{E}(\beta) = 0 \iff \chi(t) = C\psi_\beta(t).$$

Thus  $|\tilde{\chi}_N(\beta)|$  and  $\mathcal{E}(\beta)$  attain their maximum and minimum values, respectively, only when the trial return is indistinguishable from the actual return. Of course, this does not guarantee that the trial trajectory  $\beta(t)$  coincides with the actual target trajectory  $\beta_T(t)$ , since the return does not, in general, uniquely determine the target trajectory. That is, *the functionals  $\tilde{\chi}_N(\beta)$  and  $\mathcal{E}(\beta)$  are generally not one-to-one*. The class of all trajectories  $\beta$  such that  $\tilde{\chi}_N(\beta) = \tilde{\chi}_N(\beta_T)$  or, equivalently,  $\mathcal{E}(\beta) = \mathcal{E}(\beta_T)$ , represents the inherent *ambiguity* of the radar problem. A problem of obvious importance is to find outgoing signals  $\psi(t)$  which minimize this ambiguity class.

We have assumed above that the return is due to a reflection from a *single* target. If  $N$  distinct targets are involved, then we can approximate the return as a superposition

$$\psi_{\beta_1, \beta_2, \dots, \beta_N} \approx \psi_{\beta_1} + \dots + \psi_{\beta_N}. \quad (7)$$

As noted, (7) is an approximation because it ignores *multiple reflections*. Although these can often be ignored, they can also cause resonances (ringing), hence must sometimes be taken into account. This can be easily done, in principle. For example, the signal received by the doubly-reflecting path  $\alpha \rightarrow \beta_m \rightarrow \beta_n \rightarrow \gamma$  is

$$\psi_{\beta_m, \beta_n} = R_\gamma E_{\beta_n} R_{\beta_n} E_{\beta_m} R_{\beta_m} E_\alpha \psi,$$

which can be immediately converted to a triple integral by using the definitions (2) and (3). Sums of contributions from various ‘‘trial’’ scattering paths may then be matched with the actual return, defining a generalized ambiguity functional

$$\tilde{\chi}(\beta_1, \beta_2, \dots, \beta_N) \equiv \langle \chi, \psi_{\beta_1, \beta_2, \dots, \beta_N} \rangle,$$

and the Schwarz inequality may be used as in the case of a single path to optimize the match.

This method is reminiscent of *Feynman diagrams*<sup>5</sup>, where fundamental processes are represented by multiple integrals with corresponding intuitive diagrams. Because the physics is built into the formalism from the beginning through the Green functions, our model can handle such complications in a conceptually straightforward (if computationally nontrivial) way. The resemblance to Feynman diagrams is no coincidence, and the present formalism may be modified to include quantum (photonic) aspects of radar simply by using *Feynman propagators* in place of the retarded Green functions.

## REFERENCES

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