

An Algebraic Theory of Wavelets. I.
Operational Calculus and Complex Structure
(Appeared in SIAM J. Math. Anal. **23**(1992)222)

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October, 1990

Technical Reports Series #17
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ABSTRACT

In wavelet analysis, a function f is split into two parts at each iteration. The first part, Hf , represents a smoothed version of f , sampled half as frequently, while the second part, Gf , represents the detail lost by filtering through H . Although the operators H and G have very different interpretations, they exhibit a remarkable symmetry in their algebraic properties. We examine this symmetry by developing an effective, basis-independent operational calculus for wavelets and use it to show that the symmetry is due to the existence of a *complex structure*, i.e. a map J such that $J^2 = -I$ where I is the identity. This implies that the space V_α of (real) functions at the scale 2^α ($\alpha \in \mathbb{Z}$) may be regarded as a *complexification* $V_{\alpha+1}^c$ of the space $V_{\alpha+1}$ of functions at the next (coarser) scale. Roughly, the low-frequency parts Hf of the functions span the real part of $V_{\alpha+1}^c$ while their high-frequency parts Gf span the imaginary part. The map J mediates between the two and relates the corresponding operators H and G . Furthermore, at the scale $\alpha = -1$, J transforms the fundamental function ϕ associated with H into the “fundamental wavelet” ψ associated with G .

Key words: wavelets, multiscale analysis, complex structure, operational calculus.

AMS(MOS) subject classification: 16, 20, 41,42.

1. Operational Calculus

In wavelet analysis (see Daubechies [1988], Mallat [1989], Meyer [1990], Strang [1990] and the references therein), one deals with the representation of a function (“signal”) at different scales. One begins with a single real-valued function ϕ of one real variable which we take, for simplicity, to be continuous with compact support. One assumes that for some $T > 0$, the translates $\phi_n(t) \equiv \phi(t - nT)$, $n \in \mathbb{Z}$, form an orthonormal set in $L^2(\mathbb{R})$ (such functions can be easily constructed). The closure of the span of the vectors ϕ_n in $L^2(\mathbb{R})$ forms a subspace V which can be identified with $\ell^2(\mathbb{Z})$, since for a real sequence $u \equiv \{u_n\}$ we have

$$\left\| \sum_n u_n \phi_n \right\|^2 = \sum_n u_n^2. \quad (1)$$

We introduce the shift operator

$$(Sf)(t) \equiv f(t - T), \quad (2)$$

which leaves V invariant and is an orthogonal operator on $L^2(\mathbb{R})$ (we shall be dealing with *real* spaces, unless otherwise stated). A general element of V can be written uniquely as

$$\sum_n u_n S^n \phi \equiv u(S)\phi, \quad (3)$$

where $u(e^{i\xi T})$ is the square-integrable function on the unit circle ($|\xi| \leq \pi/T$) having $\{u_n\}$ as its Fourier coefficients and $u(S)$ is defined as an operator on “nice” functions (e.g., Schwartz test functions) $f(t)$ through the Fourier transform, i.e.

$$[u(S)f]^\wedge(\xi) \equiv u(e^{i\xi T})\hat{f}(\xi). \quad (4)$$

For the purpose of developing our operational calculus, we shall consider operators $u(S)$ which are *polynomials* in S and S^{-1} . These form an abelian algebra \mathcal{P} of operators on V . Moreover, it will suffice to restrict our attention to the dense subspace of finite combinations in V , i.e. to $\mathcal{P}\phi$, since our goal here is to produce an L^2 theory and this can be achieved by developing the algebraic (finite) theory and then completing in the L^2 norm. Note that the independence of the vectors ϕ_n means that $u(S)\phi = 0$ implies $u(S) = 0$. Our results could actually be extended to operators $u(S)$ with $\{u_n\} \in$

$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$, which also form an algebra since the product $u(S^{-1})w(S)$ corresponds to the convolution of the sequences $\{u_n\}$ and $\{w_n\}$. We resist the temptation.

Let us stop for a moment to discuss the “signal–processing” interpretation of $u(S)\phi$, since that is one of the motivations behind wavelet theory. It is natural to think of $u(S)\phi$ as an approximation to a function (“signal”) $f(t)$ obtained by sampling f only at $t_n = nT$. Let f_0 denote the *band–limited* function obtained from f by cutting off all frequencies ξ with $|\xi| > \pi/T$. That is, \hat{f}_0 coincides with \hat{f} for $|\xi| \leq \pi/T$ but vanishes outside this interval. The value of f_0 at t_n is then

$$f_0(nT) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} d\xi e^{-i\xi nT} \hat{f}(\xi), \quad (5)$$

which is just the Fourier coefficient of the periodic function

$$\hat{F}_0(\xi) \equiv \sum_n T f_0(nT) e^{i\xi nT} \quad (6)$$

obtained from $\hat{f}_0(\xi)$ by identifying $\xi + 2\pi/T$ with ξ . In the time domain,

$$F_0(t) = \sum_n T f_0(nT) \delta(t - nT). \quad (7)$$

This has the same form as $u(S)\phi$, if we set $u_n = T f_0(nT)$ and $\phi(t) = \delta(t)$ where δ is the Dirac distribution. Hence the usual sampling theory may be regarded as the singular case $\phi = \delta$, and then $u(S)\phi$ characterizes the band–limited approximation f_0 of f . For square–integrable ϕ , the samples u_n are no longer the values at the sharp times t_n but are *smear*ed over ϕ_n , since $u_n = \langle \phi_n, u(S)\phi \rangle$. In fact, ϕ acts as a *filter*, i.e. as a convolution operator, since $(u(S)\phi)^\wedge(\xi) = u(e^{i\xi T})\hat{\phi}(\xi)$. Roughly speaking, we may think of ϕ as giving the shape of a *pixel*.

Next, a scaled family of spaces $V_\alpha, \alpha \in \mathbb{Z}$, is constructed from V as follows. The dilation operator D , defined by

$$(Df)(t) = 2^{-1/2} f(t/2), \quad (8)$$

is orthogonal on $L^2(\mathbb{R})$. It stretches a function by a factor of 2 without altering its norm and is related to S by the commutation rules

$$DS = S^2D, \quad D^{-1}S^2 = SD^{-1}. \quad (9)$$

Hence D “squares” S while D^{-1} takes its “square root”. A repeated application of the above gives

$$D^\alpha S = S^{2^\alpha} D^\alpha, \quad \alpha \in \mathbb{Z}. \quad (10)$$

Define the spaces

$$V_\alpha = D^\alpha V, \quad (11)$$

which are closed in $L^2(\mathbb{R})$ ($V_0 \equiv V$). An orthonormal basis for V_α is given by

$$\phi_n^\alpha(t) \equiv D^\alpha S^n \phi(t) = 2^{-\alpha/2} \phi(2^{-\alpha}t - nT), \quad (12)$$

and V_α can also be identified with $\ell^2(\mathbb{Z})$. The motivation is that V_α will consist of functions containing detail only up to the scale of 2^α , which correspond to sequences $\{u_n^\alpha\}$ in $\ell^2(\mathbb{Z})$ representing samples at $t_n = 2^\alpha nT$. For this to work, we must have $V_{\alpha+1} \subset V_\alpha$ for all α . A *necessary* condition for this is that ϕ must satisfy a functional equation (taking $\alpha = -1$) of the form

$$\phi = \sum_n h_n \phi_n^{-1} = D^{-1} \sum_n h_n S^n \phi \equiv D^{-1} h(S) \phi \quad (13)$$

for some (unique) set of coefficients h_n . Since we assume that ϕ has compact support, it follows that all but a finite number of the coefficients h_n vanish, so $h(S)$ is a polynomial in S and S^{-1} , i.e. $h(S) \in \mathcal{P}$. This operator *averages*, while D^{-1} *compresses*. Hence ϕ is a fixed point of this dual action of spreading and compression. $D\phi = h(S)\phi$, called a *dilation equation*, states that the dilated pixel $D\phi$ is a linear combination of undilated pixels ϕ_n . An integration with respect to t leads to

$$\sum_n h_n = \sqrt{2}, \quad \text{or} \quad h(I) = \sqrt{2}I, \quad (14)$$

giving a constraint on the coefficients h_n (other constraints will emerge). Note that the singular case $\phi = \delta$ associated with sharp sampling satisfies the dilation equation with $h(S) = \sqrt{2}I$, where I denotes the identity operator on V . Since δ is not square-integrable, this solution does not fit into the L^2 theory considered here.

On the other hand, the coefficients h_n uniquely determine ϕ , up to a sign. For if we iterate $D^{-1}h(S) = h(S^{1/2})D^{-1}$, we obtain

$$\phi = [D^{-1}h(S)]^N \phi = \prod_{\alpha=1}^N h\left(S^{2^{-\alpha}}\right) D^{-N} \phi. \quad (15)$$

Since the Fourier transform of $D^{-N}\phi$ satisfies

$$2^{N/2} (D^{-N}\phi)^\wedge(\xi) = \hat{\phi}(2^{-N}\xi) \rightarrow \hat{\phi}(0) \text{ as } N \rightarrow \infty, \quad (16)$$

we obtain formally

$$\phi = \hat{\phi}(0) \lim_{N \rightarrow \infty} \prod_{\alpha=1}^N \left[2^{-1/2} h\left(S^{2^{-\alpha}}\right) \right] \delta. \quad (17)$$

The normalization is determined up to a sign by $\|\phi\| = 1$. In general, the function ϕ determined by $h(S)$ is highly irregular. Daubechies [1988] has classified all $h(S) \in \mathcal{P}$ which give functions ϕ possessing some regularity. The simplest (and least regular) of these is related to the classical Haar basis. It will be used throughout the paper to illustrate the various operators as they are introduced. For the general case, the actions of these operators on bases are given in the appendix.

Example 1: The Haar system.

Let ϕ be the indicator function $\chi_{[0,1)}$ for the interval $[0, 1)$. Then

$$\begin{aligned} D\phi &= \frac{1}{\sqrt{2}} \chi_{[0,2)} = \frac{1}{\sqrt{2}} (\chi_{[0,1)} + \chi_{[1,2)}) \\ &= \frac{1}{\sqrt{2}} (I + S) \phi, \end{aligned} \quad (18)$$

hence ϕ satisfies the dilation equation with $h(S) = (I + S)/\sqrt{2}$.

The next step is to introduce a “multiscale analysis” based on the sequence of spaces V_α . We shall do this in a basis-independent fashion. Since shifts and dilations are related by $DS = S^2D$, we have

$$D^{\alpha+1}u(S)\phi = D^\alpha u(S^2)D\phi = D^\alpha u(S^2)h(S)\phi. \quad (19)$$

This defines a map $H_\alpha^*: V_{\alpha+1} \rightarrow V_\alpha$, given by

$$H_\alpha^* D^{\alpha+1} u(S)\phi = D^\alpha h(S) u(S^2)\phi. \quad (20)$$

Since the two sides of this equation are actually *identical* as functions or elements of $L^2(\mathbb{R})$, H_α^* is simply the *inclusion map* which establishes $V_{\alpha+1} \subset V_\alpha$. This shows that the relation $D\phi = h(S)\phi$ is not only necessary but also *sufficient* for $V_{\alpha+1} \subset V_\alpha$. Although a vector in $V_{\alpha+1}$ is identical with its image under H_α^* as an element of $L^2(\mathbb{R})$, it is useful to distinguish between them since this permits us to use operator theory to define other useful maps, such as the adjoint $H_\alpha: V_\alpha \rightarrow V_{\alpha+1}$ of H_α^* . Since the norm on V_α is that of $L^2(\mathbb{R})$ and H_α^* is an inclusion, it follows that $H_\alpha H_\alpha^* = I_{\alpha+1}$, the identity on $V_{\alpha+1}$. In particular, H_α is *onto*; it is just the orthogonal projection from V_α to $V_{\alpha+1}$. H_α^* is interpreted as an operator which *interpolates* a vector in $V_{\alpha+1}$, representing it as the vector in V_α obtained by replacing the “pixel” ϕ with the linear combination of compressed pixels $D^{-1}h(S)\phi$. The adjoint H_α is sometimes called a “low-pass filter” because it *smooths out* the signal and re-samples it at half the sampling rate, thus cutting the frequency range in half. However, it is *not* a filter in the traditional sense since it is not a convolution operator, as will be seen below. The kernel of H_α is denoted by $W_{\alpha+1}$. It is the orthogonal complement of the image of H_α^* , i.e. of $V_{\alpha+1}$, in V_α :

$$W_{\alpha+1} \equiv \ker H_\alpha = V_\alpha \ominus H_\alpha^* V_{\alpha+1} = V_\alpha \ominus V_{\alpha+1}. \quad (21)$$

Note that H_α^* is “natural” with respect to the scale gradation, i.e.

$$H_\alpha^* D^{\alpha+1} = D^\alpha H_0^* D. \quad (22)$$

Our “home space” will be V . All our operators will enjoy the above naturality with respect to scale. Because of this property, it will generally be sufficient to work in V . Define the operator $H^*: V \rightarrow V$ by

$$H^* = H_0^* D. \quad (23)$$

We will refer to H^* as the “home version” of H_α^* . Home versions of operators will generally be denoted without subscripts. Note that while H_α^* preserves the scale (it is an inclusion map!), H^* involves a change in scale. It consists of a dilation (which spreads the sample points apart to a distance $2T$) followed by an interpolation (which

restores the sampling interval to its original value T). Thus H^* is a *zoom-in operator*! Its adjoint

$$H = D^{-1}H_0 \quad (24)$$

consists of a “filtration” by H_0 (which cuts the density of sample points by a factor of 2 without changing the scale) followed by a compression (which restores it to its previous value). H is, therefore, a *zoom-out operator*. It is related to H_α by

$$H_\alpha D^\alpha = D^{\alpha+1}H. \quad (25)$$

The filtration performed by H (which will be detailed below) represents the (possible) loss of information due to zooming out.

The operators H and H^* are essentially identical with those used by Daubechies, except for the fact that hers act on the sequences $\{u_n\}$ rather than the functions $u(S)\phi$. They are especially useful when considering iterated decomposition- and reconstruction algorithms (section 3).

To find the action of H_α , it suffices to find the action of H . Note that $H^*u(S)\phi = h(S)u(S^2)\phi$, where $u(S^2)$ is *even* in S . This will be an important observation in what follows, hence we first study the decomposition of V into its even and odd subspaces.

An arbitrary polynomial $u(S)$ in S, S^{-1} can be written uniquely as the sum of its even and odd parts,

$$\begin{aligned} u(S) &= \sum_n u_{2n}S^{2n} + \sum_n u_{2n+1}S^{2n+1} \\ &\equiv u_+(S^2) + Su_-(S^2). \end{aligned} \quad (26)$$

Define the operator E^* (for *even*) on V by

$$E^*S = S^2E^*, \quad E^*\phi = \phi. \quad (27)$$

Then

$$E^*u(S)\phi = u(S^2)\phi = \sum_n u_n\phi_{2n}. \quad (28)$$

Also define the operator O^* (for *odd*) by $O^* = SE^*$, so that

$$O^*u(S)\phi = Su(S^2)\phi = \sum_n u_n\phi_{2n+1}. \quad (29)$$

H^* is related to E^* by $H^* = h(S)E^*$. Hence to obtain H it suffices to find the adjoint E of E^* .

Lemma 1. *Let $v(S) \in \mathcal{P}$ and denote the adjoints of E^* and O^* by E and O . Then*

$$(a) \quad \begin{aligned} EE^* &= OO^* = I, \\ OE^* &= EO^* = 0, \end{aligned} \quad (30)$$

$$(b) \quad \begin{aligned} Ev(S)\phi &= v_+(S)\phi = \sum_n v_{2n}\phi_n \\ Ov(S)\phi &= v_-(S)\phi = \sum_n v_{2n+1}\phi_n, \end{aligned} \quad (31)$$

$$(c) \quad \begin{aligned} Ev(S)E^* &= v_+(S) = \frac{1}{2}D^{-1}[v(S) + v(-S)]D, \\ Ov(S)O^* &= v_+(S), \\ Ov(S)E^* &= v_-(S) = \frac{1}{2}D^{-1}S^{-1}[v(S) - v(-S)]D \\ Ev(S)O^* &= Sv_-(S) \end{aligned} \quad (32)$$

(note that (a) is a special case with $v(S) = I$), and

$$(d) \quad E^*E + O^*O = I. \quad (33)$$

Proof. For $u(S), v(S) \in \mathcal{P}$, we have

$$\begin{aligned} \langle EE^*u(S)\phi, v(S)\phi \rangle &= \langle E^*u(S)\phi, E^*v(S)\phi \rangle \\ &= \langle u(S^2)\phi, v(S^2)\phi \rangle = \langle u(S)\phi, v(S)\phi \rangle, \end{aligned} \quad (34)$$

where the last equality follows from the invariance of the inner product under $S \mapsto S^2$, i.e.

$$\langle S^{2n}\phi, S^{2m}\phi \rangle = \delta_{nm} = \langle S^n\phi, S^m\phi \rangle. \quad (35)$$

Hence $EE^* = I$, so $OO^* = ES^{-1}SE^* = I$. $EO^* = OE^* = 0$ follows from the orthogonality of even and odd functions of S (applied to ϕ). This proves (a). To show (b), note that due to the orthogonality of even and odd functions,

$$\begin{aligned}\langle E^*u(S)\phi, v(S)\phi \rangle &= \langle u(S^2)\phi, v(S)\phi \rangle = \langle u(S^2)\phi, v_+(S^2)\phi \rangle \\ &= \langle E^*u(S)\phi, E^*v_+(S)\phi \rangle = \langle u(S)\phi, v_+(S)\phi \rangle,\end{aligned}\tag{36}$$

where we have used (a). This proves the first equation in (b). The second follows from $O = ES^{-1}$ and $S^{-1}v(S) = v_-(S^2) + S^{-1}v_+(S^2)$. To prove (c), note that $u(S^2)E^* = E^*u(S)$ and $Su(S^2)E^* = O^*u(S)$, hence

$$\begin{aligned}Ev(S)E^* &= E(v_+(S^2) + Sv_-(S^2))E^* = EE^*v_+(S) + EO^*v_-(S) = v_+(S), \\ Ov(S)O^* &= ES^{-1}v(S)SE^* = v_+(S), \\ Ov(S)E^* &= O(v_+(S^2) + Sv_-(S^2))E^* = OE^*v_+(S) + EE^*v_-(S) = v_-(S), \\ Ev(S)O^* &= E(v_+(S^2) + Sv_-(S^2))SE^* = EO^*v_+(S) + EE^*Sv_-(S) = Sv_-(S).\end{aligned}\tag{37}$$

Lastly, (d) follows from

$$\begin{aligned}E^*Ev(S)\phi + O^*Ov(S)\phi &= E^*v_+(S)\phi + O^*v_-(S)\phi \\ &= v_+(S^2)\phi + Sv_-(S^2)\phi = v(S)\phi. \quad \blacksquare\end{aligned}\tag{38}$$

Remark. The algebraic structure above is characteristic of orthogonal decompositions and will be met again in our discussion of low- and high-frequency filters. E^*E and O^*O are the projection operators to the subspaces of even and odd functions of S (applied to ϕ),

$$V^e \equiv \overline{\{v(S^2)\phi \mid v(S) \in \mathcal{P}\}}, \quad V^o \equiv \overline{\{Sv(S^2)\phi \mid v(S) \in \mathcal{P}\}},\tag{39}$$

and

$$V = V^e \oplus V^o.\tag{40}$$

This decomposition will play an important role in the sequel.

Proposition 2. The maps $H: V \rightarrow V$ and $H_\alpha: V_\alpha \rightarrow V_{\alpha+1}$ are given by

$$\begin{aligned}
Hu(S)\phi &= Eh(S^{-1})u(S)\phi \\
&= [h_+(S^{-1})u_+(S) + h_-(S^{-1})u_-(S)] \phi, \\
H_\alpha D^\alpha u(S)\phi &= D^{\alpha+1} E h(S^{-1})u(S)\phi \\
&= D^{\alpha+1} [h_+(S^{-1})u_+(S) + h_-(S^{-1})u_-(S)] \phi.
\end{aligned} \tag{41}$$

Proof. Since $H^* = h(S) E^*$, it follows that $H = E h(S^{-1})$ and $H_\alpha D^\alpha = D^{\alpha+1} H = D^{\alpha+1} E h(S^{-1})$. ■

Example 2: The Haar system, continued.

For $h(S) = (I + S)/\sqrt{2}$ and $\phi = \chi_{[0,1]}$, we have $h_+(S) = h_-(S) = 1/\sqrt{2}$. Hence

$$\begin{aligned}
H^* \phi_n &= h(S) \phi_{2n} = \frac{1}{\sqrt{2}} (\phi_{2n} + \phi_{2n+1}) \\
H \phi_n &= \frac{1}{\sqrt{2}} E (I + S^{-1}) S^n \phi = \frac{1}{\sqrt{2}} E (S^n + S^{n-1}) \phi \\
&= \frac{1}{\sqrt{2}} \begin{cases} S^{n/2} \phi, & n \text{ even} \\ S^{(n-1)/2} \phi, & n \text{ odd} \end{cases} \\
&= \frac{1}{\sqrt{2}} \phi_{[n/2]}.
\end{aligned} \tag{42}$$

2. Complex Structure

Up to this point, it could be argued, nothing extraordinary has happened. We have a filter which, when applied repeatedly, gives rise to a nested sequence of subspaces V_α . However, the next step is quite surprising and underlies much of the interest wavelets have generated. It is desirable to record the information lost at each stage of filtering, i.e., that part of the signal residing in the orthogonal complement $W_{\alpha+1}$ of $V_{\alpha+1}$ in V_α . The orthogonal decomposition $V_\alpha = V_{\alpha+1} \oplus W_{\alpha+1}$ is described by filters H_α and G_α , where H_α is as above and G_α extracts high-frequency information. For this reason, H_α and G_α obey a set of algebraic relations similar to those satisfied by E and O above. What is quite remarkable is that there exists a vector ψ in V_{-1} which is related to the spaces W_α and the maps G_α in a way almost totally symmetric to the way ϕ is related

to V_α and H_α . This is *not* merely a consequence of the orthogonal decomposition but is somehow related to the fact that $V_{\alpha+1}$ is “half” of V_α , due to the doubling of the sampling interval upon dilation, as expressed by the commutation relation $DS = S^2D$. However, the precise *reason* for this symmetry has not been entirely clear. The usual constructions are somewhat involved and do not appear to shed much light on this question. It was this puzzle which motivated the present work. As an answer, we propose the following new construction. Begin by defining a *complex structure* on V , i.e. a map $J: V \rightarrow V$ such that $J^2 = -I$. (To illustrate this concept, consider the complex plane \mathbb{C} as the real space \mathbb{R}^2 . Then multiplication by the unit imaginary i is represented by a real 2×2 matrix whose square is $-I$.) J is defined by giving its commutation rule with respect to the shift and its action on ϕ :

$$JS = -S^{-1}J, \quad J\phi = \varepsilon(S)\phi, \quad (1)$$

where $\varepsilon(S)$ is an as yet undetermined function. It follows that for $u(S) \in \mathcal{P}$,

$$Ju(S)\phi = \varepsilon(S)u(-S^{-1})\phi. \quad (2)$$

We further require that J preserve the inner product, i.e. that $J^*J = I$. Combined with $J^2 = -I$, this gives $J^* = -J$. That is, J will behave like multiplication by i also with respect to the inner product, giving it an interpretation as a *Hermitian* inner product.

In order to study J , we first define two simpler operators C and M as follows.

$$\begin{aligned} CS &= S^{-1}C, & C\phi &= \phi \\ MS &= -SM, & M\phi &= \phi. \end{aligned} \quad (3)$$

Note that $CM = MC$ and that $C^* = C$ and $M^* = M$, since

$$\begin{aligned} \langle Cu(S)\phi, v(S)\phi \rangle &= \langle u(S^{-1})\phi, v(S)\phi \rangle = \langle v(S)\phi, u(S^{-1})\phi \rangle \\ &= \langle u(S)\phi, v(S^{-1})\phi \rangle = \langle u(S)\phi, Cv(S)\phi \rangle, \end{aligned} \quad (4)$$

where we have used the symmetry of the inner product and $u(S)^* = u(S^{-1})$ (both of which depend on the reality of V), and

$$\begin{aligned} \langle Mu(S)\phi, v(S)\phi \rangle &= \langle u(-S)\phi, v(S)\phi \rangle = \langle u(S)\phi, v(-S)\phi \rangle \\ &= \langle u(S)\phi, Mv(S)\phi \rangle, \end{aligned} \quad (5)$$

where the invariance of the inner product under $S \mapsto -S$ was used. ($C^* = C$ could have been proved more simply by noting that the inner product is invariant under $S \mapsto S^{-1}$, which follows from the orthogonality of the ϕ_n 's; however, this orthogonality does not appear to be a fundamental feature of the theory, so we avoid it whenever possible.) Since C and M are also involutions, i.e.

$$C^2 = M^2 = I, \quad (6)$$

it follows that they are orthogonal operators. Hence they represent *symmetries*, which makes them important in themselves, especially in the abstract context where one begins with an algebra and constructs a representation (see the remark at the end of section 3). In fact, the orthogonal decomposition $V = V^e \oplus V^o$ is nothing but the spectral decomposition associated with M , since V^e and V^o are the eigenspaces of M with eigenvalues 1 and -1 , respectively. C has a simple interpretation as a *conjugation operator*, since for $u(S) \in \mathcal{P}$,

$$Cu(S)C = u(S^{-1}) = u(S)^*. \quad (7)$$

In terms of C and M ,

$$J = \varepsilon(S)CM. \quad (8)$$

Proposition 3. *The conditions $J^* = -J$ and $J^2 = -I$ hold if and only if $\varepsilon(S)$ satisfies*

$$\varepsilon(-S) = -\varepsilon(S), \quad \varepsilon(S^{-1})\varepsilon(S) = 1. \quad (9)$$

Proof. We have

$$J^* = MC\varepsilon(S^{-1}) = M\varepsilon(S)C = \varepsilon(-S)MC = \varepsilon(-S)CM, \quad (10)$$

hence $J^* = -J$ if and only if $\varepsilon(-S) = -\varepsilon(S)$. Assume this to be the case. Then

$$J^2 = \varepsilon(S)CM\varepsilon(S)CM = \varepsilon(S)\varepsilon(-S^{-1}) = -\varepsilon(S)\varepsilon(S^{-1}), \quad (11)$$

hence $J^2 = -I$ if and only if $\varepsilon(S^{-1})\varepsilon(S) = I$. \blacksquare

Remarks.

1. J is determined only up to the orthogonal mapping $\varepsilon(S)$. This corresponds to a similar freedom in the standard approach to wavelet theory, where a factor $e^{i\lambda(\xi)}$ in Fourier space relates the functions $H(\xi)$ and $G(\xi)$ associated with the operators H and G (Daubechies [1988], p. 943, where $T = 1$). The relation between $\varepsilon(S)$ and $\lambda(\xi)$ is given in the Appendix.
2. The simplest examples of a complex structure are given by choosing

$$\varepsilon(S) = \pm S^{2p+1}, \quad p \in \mathbb{Z}. \quad (12)$$

More interesting examples can be obtained by enlarging \mathcal{P} to a topological algebra, for example allowing $u(S)$ with $\{u_n\} \in \ell^1(\mathbb{Z})$.

3. The above proof used the symmetry of the inner product. Later we shall complexify our spaces and the inner product becomes Hermitian. However, this proof easily extends to the complex case (when transposing, also take the complex conjugate). C then becomes \mathbb{C} -*antilinear* and is interpreted as Hermitian conjugation.

At an arbitrary scale α , define maps $J_\alpha: V_\alpha \rightarrow V_\alpha$ by naturality, i.e.

$$J_\alpha D^\alpha = D^\alpha J, \quad (13)$$

which implies that $J_\alpha^2 = -I_\alpha$ and $J_\alpha^* = -J_\alpha$. J_α is related to S by

$$\begin{aligned} S^{2\alpha} J_\alpha D^\alpha &= S^{2\alpha} D^\alpha J = D^\alpha S J = -D^\alpha J S^{-1} \\ &= -J_\alpha D^\alpha S^{-1} = -J_\alpha S^{-2\alpha} D^\alpha, \end{aligned} \quad (14)$$

showing that

$$S^{2\alpha} J_\alpha = -J_\alpha S^{-2\alpha}. \quad (15)$$

In particular, note that $S^{1/2} J_{-1} = -J_{-1} S^{-1/2}$, hence

$$S J_{-1} = +J_{-1} S^{-1}. \quad (16)$$

We are now in a position to construct the basic wavelet ψ , the spaces W_α and an appropriate set of high-frequency filters in a way which will make the symmetry with

ϕ , V_α and H_α quite clear. Consider the *restriction* of J_α to the subspace $V_{\alpha+1}$ of V_α , i.e. the map $K_\alpha^*: V_{\alpha+1} \rightarrow V_\alpha$ defined by

$$K_\alpha^* = J_\alpha H_\alpha^*. \quad (17)$$

K_α^* is natural with respect to the scale gradation, and its home version will, as usual, be denoted by $K^* \equiv K_0^* D = JH^*$. It will turn out that its adjoint K is essentially equivalent to the usual filter G (to be introduced below) but is more natural from the point of view of the complex structure. Define the vector $\psi \in V_{-1}$ by

$$\psi = K_{-1}^* \phi = J_{-1} D^{-1} h(S) \phi = D^{-1} J h(S) \phi \equiv D^{-1} g(S) \phi, \quad (18)$$

where the function

$$g(S) = \varepsilon(S) h(-S^{-1}) \quad (19)$$

will play a similar role for the high-frequency components as does $h(S)$ for the low-frequency components. Namely, $g(S)$ is a “differencing operator”, just as $h(S)$ is an averaging operator. (We will see below that for the Haar system, $g(S) = (I - S)/\sqrt{2}$.) For $w(S) \in \mathcal{P}$, we have

$$K^* w(S) \phi = JH^* w(S) \phi = Jh(S) w(S^2) \phi = g(S) w(S^{-2}) \phi = g(S) E^* C w(S) \phi, \quad (20)$$

hence

$$K^* = g(S) E^* C = g(S) C E^*. \quad (21)$$

Proposition 4. *The adjoints of K^* and K_α^* are given by*

$$\begin{aligned} K v(S) \phi &= ECg(S^{-1})v(S)\phi = Ev(S^{-1})D\psi, \\ K_\alpha D^\alpha v(S) \phi &= D^{\alpha+1} ECg(S^{-1})v(S)\phi = D^{\alpha+1} Ev(S^{-1})D\psi. \end{aligned} \quad (22)$$

Proof. Since $K = ECg(S^{-1})$ and $K_\alpha D^\alpha = D^{\alpha+1} K$, we have

$$K v(S) \phi = ECg(S^{-1})v(S)\phi = Eg(S)v(S^{-1})\phi = Ev(S^{-1})D\psi \quad (23)$$

and

$$K_\alpha D^\alpha v(S)\phi = D^{\alpha+1} E g(S^{-1})v(S)\phi = D^{\alpha+1} E v(S^{-1})D\psi. \quad \blacksquare \quad (24)$$

Example 3: The Haar system, continued.

Returning to $h(S) = (I + S)/\sqrt{2}$ and choosing $\varepsilon(S) = -S$, we have

$$\begin{aligned} J\phi_n &= -S(-S^{-1})^n \phi = (-1)^{1-n} \phi_{1-n} \\ g(S) &= -S \frac{1}{\sqrt{2}} (I - S^{-1}) = \frac{1}{\sqrt{2}} (I - S) \\ \psi &= D^{-1} \frac{1}{\sqrt{2}} (I - S) \phi = \frac{1}{\sqrt{2}} (\chi_{[0,1/2)} - \chi_{[1/2,1)}) \\ K^* \phi_n &= JH^* \phi_n = \frac{1}{\sqrt{2}} J(\phi_{2n} + \phi_{2n+1}) \\ &= \frac{1}{\sqrt{2}} (\phi_{-2n} - \phi_{1-2n}) \\ K \phi_n &= E S^{-n} D\psi = \frac{1}{\sqrt{2}} E S^{-n} (I - S) \phi \\ &= \frac{1}{\sqrt{2}} E (S^{-n} - S^{1-n}) \phi \\ &= \frac{1}{\sqrt{2}} \begin{cases} S^{-n/2} \phi, & n \text{ even} \\ S^{(1-n)/2} \phi, & n \text{ odd} \end{cases} \\ &= \frac{(-1)^n}{\sqrt{2}} \phi_{-[n/2]}. \end{aligned} \quad (25)$$

It can be easily checked that the functions $\psi_n^\alpha \equiv D^\alpha S^n \psi$ ($\alpha, n \in \mathbb{Z}$) are mutually orthogonal. They form the *Haar basis* of $L^2(\mathbb{R})$.

Note: The “time-reversal” associated with K and K^* (i.e., $n \rightarrow -n$) is due to the presence of C in J . It is harmless, since ultimately it is only KK^* and K^*K that count. However, it can be removed by replacing K_α^* and K^* by

$$\begin{aligned} K'_\alpha{}^* &\equiv J_\alpha H_\alpha^* C = K_\alpha^* C \\ K'^* &\equiv K_0'^* D = K^* C. \end{aligned} \quad (26)$$

For then

$$\begin{aligned}
K'K'^* &= I, & K'^*K' &= K^*K \\
K'H^* &= 0, & HK'^* &= 0,
\end{aligned} \tag{27}$$

with similar formulas for H_α and K'_α . Hence H and K' lead to an orthogonal decomposition of V equivalent to that given by H and K . Furthermore, K' does not reverse time. For the Haar system,

$$\begin{aligned}
K'^*\phi_n &= K^*C\phi_n = K^*\phi_{-n} = \frac{1}{\sqrt{2}}(\phi_{2n} - \phi_{2n+1}) \\
K'\phi_n &= CK\phi_n = \frac{(-1)^n}{\sqrt{2}}C\phi_{-[n/2]} \\
&= \frac{(-1)^n}{\sqrt{2}}\phi_{[n/2]}.
\end{aligned} \tag{28}$$

We prefer K_α and K because they are “cleaner” with respect to the complex structure. For example, the complex decomposition– and reconstruction algorithm given in section 3 is less natural if K'_α and K' are used.

Proposition 5. *The pairs of operators $\{H, K\}$ and $\{H_\alpha, K_\alpha\}$ satisfy*

$$\begin{aligned}
HH^* &= Eh(S^{-1})h(S)E^* = I, \\
KK^* &= Eg(S^{-1})g(S)E^* = I, \\
HK^* &= Eh(S^{-1})g(S)E^*C = 0, \\
KH^* &= CEg(S^{-1})h(S)E^* = 0.
\end{aligned} \tag{29}$$

and

$$H_\alpha H_\alpha^* = K_\alpha K_\alpha^* = I_{\alpha+1}, \quad H_\alpha K_\alpha^* = K_\alpha H_\alpha^* = 0. \tag{30}$$

Proof. (Note that $H_\alpha H_\alpha^* = I_{\alpha+1}$ has already been shown; it is included here for completeness, since it belongs with the other identities.) The first equation follows from $H^* = H_0^*D$ and $H_0^*H_0 = I$. The second follows from the first and $K^* = JH^*$, since $J^*J = I$. The last two equations follow from lemma 1, since $h(S^{-1})g(S)$ and $g(S^{-1})h(S)$ are odd functions, hence their even parts vanish. This proves the identities for H and K . The other identities follow by naturality. ■

Proposition 6. *The pairs $\{H, K\}$ and $\{H_\alpha, K_\alpha\}$ give orthogonal decompositions of V and V_α . We have*

(a)

$$\begin{aligned} HV &= KV = V \\ H^*V &= V_1, \quad K^*V = W_1 \\ H^*H + K^*K &= I, \end{aligned} \tag{31}$$

(b)

$$\begin{aligned} H_\alpha V_\alpha &= K_\alpha V_\alpha = V_{\alpha+1} \\ H_\alpha^* V_{\alpha+1} &= V_{\alpha+1}, \quad K_\alpha^* V_{\alpha+1} = W_{\alpha+1} \\ H_\alpha^* H_\alpha + K_\alpha^* K_\alpha &= I_\alpha. \end{aligned} \tag{32}$$

Proof. We have

$$HV = D^{-1}H_0V = D^{-1}V_1 = V, \tag{33}$$

and since $K = -HJ$, it follows that $KV = HJV = HV = V$. Also

$$H^*V = H_0^*DV = H_0^*V_1 = V_1. \tag{34}$$

$KK^* = I$ and $HK^* = 0$ (proposition 4) imply that K^* is injective and its range is orthogonal to that of H^* , i.e. to V_1 . Hence $K^*V \subset W_1$. To show that $K^*V = W_1$, let $u(S) \in \mathcal{P}$. We need to find $v(S), w(S) \in \mathcal{P}$ such that

$$u(S)\phi = H^*v(S)\phi + K^*w(S)\phi = h(S)v(S^2)\phi + g(S)w(S^{-2})\phi \tag{35}$$

or, equivalently, dropping ϕ ,

$$u(S) = h(S)v(S^2) + g(S)w(S^{-2}). \tag{36}$$

Use lemma 1 to decompose this equation into its even and odd parts:

$$\begin{aligned} u_+(S) &= Eu(S)E^* = E [h(S)v(S^2) + g(S)w(S^{-2})] E^* \\ &= h_+(S)v(S) + g_+(S)w(S^{-1}), \\ u_-(S) &= Ou(S)E^* = O [h(S)v(S^2) + g(S)w(S^{-2})] E^* \\ &= h_-(S)v(S) + g_-(S)w(S^{-1}), \end{aligned} \tag{37}$$

which can be written in matrix form as

$$\begin{bmatrix} u_+(S) \\ u_-(S) \end{bmatrix} = \begin{bmatrix} h_+(S) & g_+(S) \\ h_-(S) & g_-(S) \end{bmatrix} \begin{bmatrix} v(S) \\ w(S^{-1}) \end{bmatrix} \equiv U(S) \begin{bmatrix} v(S) \\ w(S^{-1}) \end{bmatrix}. \quad (38)$$

But proposition 5 is precisely the statement that the matrix $U(S)$ is *unitary* i.e., $U(S)^*U(S) = I$. Multiplying by $U(S)^*$, we obtain the unique solution

$$\begin{bmatrix} v(S) \\ w(S^{-1}) \end{bmatrix} = \begin{bmatrix} h_+(S^{-1}) & h_-(S^{-1}) \\ g_+(S^{-1}) & g_-(S^{-1}) \end{bmatrix} \begin{bmatrix} u_+(S) \\ u_-(S) \end{bmatrix}, \quad (39)$$

which shows that $V \equiv V_1 \oplus W_1 = H^*V \oplus K^*V$. Applying H and K to eq. (35) gives

$$v(S)\phi = Hu(S)\phi, \quad w(S)\phi = Ku(S)\phi, \quad (40)$$

which proves that $H^*H + K^*K = I$ as claimed. For the range of K_α^* we have $K_\alpha^*V_{\alpha+1} = K_\alpha^*D^{\alpha+1}V = D^\alpha K^*V = D^\alpha W_1 = W_{\alpha+1}$. Finally,

$$H_\alpha^*H_\alpha + K_\alpha^*K_\alpha = D^\alpha(H^*H + K^*K)D^{-\alpha} = I_\alpha. \quad \blacksquare \quad (41)$$

We now construct the usual “high–frequency filters” G_α . First note that elements in $W_1 \equiv K^*V$ can be written in the form

$$K^*w(S)\phi = g(S)w(S^{-2})\phi = w(S^{-2})D\psi = Dw(S^{-1})\psi. \quad (42)$$

It follows that ψ is a “basic wavelet”, i.e. that

$$W_\alpha = \overline{D^\alpha \mathcal{P}\psi}, \quad (43)$$

and the vectors

$$\psi_n^\alpha \equiv D^\alpha S^n \psi = D^{\alpha-1} K^* S^{-n} \phi = D^{-1} K_\alpha^* \phi_{-n}^{\alpha+1} \quad (44)$$

form an orthonormal basis for W_α .

Since $K_0^*: V_1 \rightarrow V$ is injective and its range is W_1 , it can be factored uniquely as

$$K_0^* = G_0^* R_1^*, \quad (45)$$

where $R_1^*: V_1 \rightarrow W_1$ is an isomorphism and $G_0^*: W_1 \rightarrow V$ is the inclusion map. From

$$K_0^* D w(S) \phi = D w(S^{-1}) \psi = g(S) w(S^{-2}) \phi \quad (46)$$

we read off

$$\begin{aligned} R_1^* D w(S) \phi &= D w(S^{-1}) \psi, \\ G_0^* D w(S^{-1}) \psi &= g(S) w(S^{-2}) \phi. \end{aligned} \quad (47)$$

For the home versions, we have

$$K^* = G_0^* R_1^* D = G_0^* D R^* = G^* R^*, \quad (48)$$

where

$$R^*: V \equiv V_0 \rightarrow W \equiv W_0 \quad (49)$$

is an isomorphism and $G^*: W \rightarrow V$, though injective, is not an inclusion map. (This is the price for working with the home versions, which do not preserve the scale.) Note that $R^* R = I_W$ and $R R^* = I_V$, hence $K = R G$ implies that $G = R^* K$ and

$$\begin{aligned} G G^* &= R^* K K^* R = I_W, \\ G H^* &= R^* K H^* = 0, \\ G^* G &= K^* R R^* K = K^* K, \end{aligned} \quad (50)$$

hence $H^* H + G^* G = I$. Therefore H and G give an orthogonal decomposition of V which is equivalent to that given by H and K .

Defining $R_\alpha^*: V_\alpha \rightarrow W_\alpha$ and $G_\alpha^*: W_{\alpha+1} \rightarrow V_\alpha$ by $R_\alpha^* D^\alpha = D^\alpha R^*$ and $G_\alpha^* D^{\alpha+1} = D^\alpha G^*$, we get a graded family of filters G_α related to K_α by

$$K_\alpha^* = G_\alpha^* R_{\alpha+1}^*. \quad (51)$$

The operators G_α are, in fact, the usual high-frequency filters. For the latter are defined analogously to H_α , namely by substituting $g(S)\phi$ for the dilated wavelet $D\psi$:

$$G_\alpha^* D^{\alpha+1} u(S) \psi = G_\alpha^* D^\alpha u(S^2) D \psi = D^\alpha g(S) u(S^2) \phi = D^\alpha G^* u(S) \psi. \quad (52)$$

The orthogonal decomposition of V given by H and G induces an orthogonal decomposition of V_α by H_α and G_α which is, in fact, the usual wavelet decomposition and is equivalent to the one given by H_α and K_α in proposition 6.

Example 4: The Haar system, final visit.

For the Haar system, eqs. (47) and (48) give

$$\begin{aligned}
R^* \phi_n &= D^{-1} R_1^* D \phi_n = S^{-n} \psi = \psi_{-n} \\
G^* \psi_n &= G_0^* D \psi_n = \frac{1}{\sqrt{2}} (I - S) \phi_{2n} = \frac{1}{\sqrt{2}} (\phi_{2n} - \phi_{2n+1}) \\
G \phi_n &= R^* K \phi_n = \frac{(-1)^n}{\sqrt{2}} R^* \phi_{-[n/2]} \\
&= \frac{(-1)^n}{\sqrt{2}} \psi_{[n/2]}.
\end{aligned} \tag{53}$$

Remark. We have stated that K_α is more natural than G_α from the point of view of the symmetry associated with J_α . The reason is that both H_α and K_α map V_α to $V_{\alpha+1}$, whereas G_α maps V_α to $W_{\alpha+1}$. This symmetry is reflected by the simple relation $K_\alpha^* = J_\alpha H_\alpha^*$, whereas the relation

$$G_\alpha^* = J_\alpha H_\alpha^* R_{\alpha+1} \tag{54}$$

is somewhat more complicated. A more concrete dividend of this symmetry will appear in the next section, where the complex combinations $H_\alpha^* \pm iK_\alpha^*$ will be considered. The corresponding combinations $H_\alpha^* \pm iG_\alpha^*$ do not make sense, as the two operators have different domains.

We now prove an identity which will be useful later.

Proposition 7.

$$\begin{aligned}
K^* H - H^* K &= J, \\
K_\alpha^* H_\alpha - H_\alpha^* K_\alpha &= J_\alpha.
\end{aligned} \tag{55}$$

Proof. For $u(S)\phi = H^*v(S)\phi + K^*w(S)\phi \in V$, we have

$$\begin{aligned}
(K^* H - H^* K)u(S)\phi &= K^*v(S)\phi - H^*w(S)\phi \\
&= JH^*v(S)\phi + JK^*w(S)\phi = Ju(S)\phi.
\end{aligned} \tag{56}$$

The identity at arbitrary scale follows from naturality:

$$(K_\alpha^* H_\alpha - H_\alpha^* K_\alpha) D^\alpha = D^\alpha (K^* H - H^* K) = D^\alpha J = J_\alpha D^\alpha. \quad \blacksquare \quad (57)$$

It is natural to wonder whether the complex structures J_α extend to define a “global” complex structure on $L^2(\mathbb{R})$. We now show that this is not the case.

Proposition 8. *The complex structure J_α on V_α is not an extension of $J_{\alpha+1}$.*

Proof. The statement that J_α is an extension of $J_{\alpha+1}$ means that $J_{\alpha+1}$ is the restriction of J_α to $V_{\alpha+1}$, i.e.,

$$K_\alpha^* \equiv J_\alpha H_\alpha^* = H_\alpha^* J_{\alpha+1}. \quad (58)$$

If this were true, then left-multiplication by H_α would imply

$$0 = H_\alpha K_\alpha^* = H_\alpha H_\alpha^* J_{\alpha+1} = J_{\alpha+1}, \quad (59)$$

which contradicts $J_{\alpha+1}^2 = -I_{\alpha+1}$. \blacksquare

3. Complex Decomposition and Reconstruction

The decomposition/reconstruction algorithm of the last section can be iterated, and when repeated indefinitely gives a unique representation of any function $f \in L^2(\mathbb{R})$ as an L^2 -convergent infinite orthogonal sum of “detail” functions at finer and finer scales. We develop the algorithm formally in the home version, which will be seen to be much more convenient. For a rigorous treatment, see Daubechies [1988b]. Given any $D^\alpha v^\alpha(S)\phi \in V_\alpha$, write

$$v^\alpha \equiv v^\alpha(S)\phi \in V \quad (1)$$

for brevity. Since

$$\begin{aligned} I &= K^* K + H^* H = K^* K + H^* (K^* K + H^* H) H = \dots \\ &= \sum_{\beta=1}^N (H^*)^{\beta-1} K^* K H^{\beta-1} + (H^*)^N H^N, \end{aligned} \quad (2)$$

we have

$$\begin{aligned}
v^\alpha &= \sum_{\beta=1}^N (H^*)^{\beta-1} K^* K H^{\beta-1} v^\alpha + (H^*)^N H^N v^\alpha \\
&= \sum_{\beta=1}^N (H^*)^{\beta-1} K^* w^{\alpha+\beta} + (H^*)^N v^{\alpha+N},
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
w^{\alpha+\beta} &\equiv K H^{\beta-1} v^\alpha \\
v^{\alpha+N} &\equiv H^N v^\alpha.
\end{aligned} \tag{4}$$

The vector $v^{\alpha+N}$ represents an N -fold smoothed version of v^α , while $w^{\alpha+\beta}$ represents the detail filtered out at β -th iteration. The terms in the above sum are orthogonal, since for $\gamma > \beta$ we have

$$\begin{aligned}
&\langle (H^*)^{\beta-1} K^* w^{\alpha+\beta}, (H^*)^{\gamma-1} K^* w^{\alpha+\gamma} \rangle \\
&= \langle w^{\alpha+\beta}, K (H^*)^{\gamma-\beta} K^* w^{\alpha+\beta} \rangle = 0.
\end{aligned} \tag{5}$$

To see what this expansion means in terms of the scale-preserving filters, apply D^α and use naturality (see appendix):

$$\begin{aligned}
D^\alpha v^\alpha &= \sum_{\beta=1}^N H_\alpha^* H_{\alpha+1}^* \cdots H_{\alpha+\beta-2}^* K_{\alpha+\beta-1}^* D^{\alpha+\beta} w^{\alpha+\beta} \\
&\quad + H_\alpha^* H_{\alpha+1}^* \cdots H_{\alpha+N-1}^* D^{\alpha+N} v^{\alpha+N}.
\end{aligned} \tag{6}$$

This formula shows the advantage of the home versions of the filters, which “zoom” in and out to get the detail at any desirable scale and can therefore be used repeatedly without changing operators.

It can be shown that $v^{\alpha+N} \rightarrow 0$ in $L^2(\mathbb{R})$ as $N \rightarrow \infty$, which gives the orthogonal decomposition

$$D^\alpha v^\alpha = D^\alpha \sum_{\beta=1}^{\infty} (H^*)^{\beta-1} K^* w^{\alpha+\beta}. \tag{7}$$

Since

$$L^2(\mathbb{R}) = \overline{\bigcup_{\alpha \in \mathbb{Z}} V_\alpha}, \quad (8)$$

any $L^2(\mathbb{R})$ function can be approximated as accurately as desired in the form of eq. (7). One need only choose a “cut-off” scale α , which means that all detail finer than 2^α will be ignored. Moreover, since

$$W_{\alpha+1} \subset V_\alpha \perp W_\alpha, \quad (9)$$

the above gives the orthogonal decomposition

$$L^2(\mathbb{R}) = \overline{\bigoplus_{\alpha \in \mathbb{Z}} W_\alpha}. \quad (10)$$

Let us now look at the reconstruction and decomposition formulas in light of the complex structure. A single iteration gives

$$v^\alpha = H^* v^{\alpha+1} + K^* w^{\alpha+1} = H^* v^{\alpha+1} + JH^* w^{\alpha+1}, \quad (11)$$

where

$$v^{\alpha+1} = H v^\alpha, \quad w^{\alpha+1} = K v^\alpha = -H J v^\alpha. \quad (12)$$

Since J is like multiplication by $\pm i$, let us define the complex conjugate pairs of vectors

$$\begin{aligned} \zeta^\alpha &= \frac{v^\alpha + i w^\alpha}{\sqrt{2}} \\ \bar{\zeta}^\alpha &= \frac{v^\alpha - i w^\alpha}{\sqrt{2}}, \end{aligned} \quad (13)$$

where the $\sqrt{2}$ in the denominators is for later convenience. ζ^α and $\bar{\zeta}^\alpha$ belong to the *complexification* of V , i.e. to

$$V^c = V \oplus iV = \mathbb{C} \otimes V. \quad (14)$$

We endow V^c with the Hermitian inner product obtained from V by extending \mathbb{C} -linearly in the *second* factor and antilinearly in the first factor (this is the convention used in the physics literature). Note that under this inner product, iV is not orthogonal to V , hence the direct sum $V \oplus iV$ is *not* an orthogonal decomposition. We now extend

all operators on V to V^c by \mathbb{C} -linearity, denoting the extensions by the same symbols. Substituting

$$v^{\alpha+1} = \frac{\zeta^{\alpha+1} + \bar{\zeta}^{\alpha+1}}{\sqrt{2}}, \quad w^{\alpha+1} = \frac{\zeta^{\alpha+1} - \bar{\zeta}^{\alpha+1}}{\sqrt{2}i} \quad (15)$$

into the reconstruction formula, we obtain

$$v^\alpha = Z^* \zeta^{\alpha+1} + \bar{Z}^* \bar{\zeta}^{\alpha+1} = 2 \Re(Z^* \zeta^{\alpha+1}), \quad (16)$$

where the operators $Z^*, \bar{Z}^*: V^c \rightarrow V^c$ are defined by

$$Z^* = \frac{H^* - iK^*}{\sqrt{2}}, \quad \bar{Z}^* = \frac{H^* + iK^*}{\sqrt{2}}. \quad (17)$$

Therefore

$$Z = \frac{H + iK}{\sqrt{2}} = \frac{H - iHJ}{\sqrt{2}} = H \left(\frac{I - iJ}{\sqrt{2}} \right). \quad (18)$$

The operators

$$P^\pm = \frac{I \mp iJ}{2} \quad (19)$$

are the orthogonal projections in V^c to the eigenspaces of J with eigenvalues $\pm i$, since

$$\begin{aligned} JP^\pm &= \pm iP^\pm, \\ (P^\pm)^* &= P^\pm = (P^\pm)^2. \end{aligned} \quad (20)$$

We have the orthogonal decomposition

$$V^c = V^+ \oplus V^-, \quad V^\pm \equiv P^\pm V. \quad (21)$$

The above shows that the operator $Z: V^c \rightarrow V^c$ and its adjoint satisfy

$$Z = \sqrt{2}HP^+, \quad Z^* = \sqrt{2}P^+H^*. \quad (22)$$

Since H^* is injective, it follows that the range of Z^* is V^+ and the kernel of Z is V^- . Similarly,

$$\bar{Z} = \sqrt{2}HP^-, \quad \bar{Z}^* = \sqrt{2}P^-H^*, \quad (23)$$

so the range of \bar{Z}^* is V^- and the kernel of \bar{Z} is V^+ .

Proposition 9. *The operators Z and \bar{Z} satisfy*

$$\begin{aligned} ZZ^* &= I, & \bar{Z}\bar{Z}^* &= I, \\ Z\bar{Z}^* &= 0, & \bar{Z}Z^* &= 0, \\ Z^*Z &= P^+, & \bar{Z}^*\bar{Z} &= P^-. \end{aligned} \tag{24}$$

Proof. By proposition 5, we have

$$\begin{aligned} ZZ^* &= 2HP^+P^+H^* = 2HP^+H^* \\ &= H(I - iJ)H^* = HH^* - iHK^* = I, \end{aligned} \tag{25}$$

and

$$Z\bar{Z}^* = 2HP^+P^-H^* = 0. \tag{26}$$

Furthermore, by propositions 6 and 7,

$$\begin{aligned} 2Z^*Z &= (H^* - iK^*)(H + iK) \\ &= (H^*H + K^*K) + i(H^*K - K^*H) \\ &= I - iJ = 2P^+. \end{aligned} \tag{27}$$

The other equalities follow from complex conjugation, which exchanges Z and \bar{Z} . ■

Proposition 9 gives a new, *complex* decomposition– and reconstruction algorithm. Given $\zeta^\alpha \in V^c$, define $\zeta^{\alpha+1}, \chi^{\alpha+1} \in V^c$ by

$$\begin{aligned} \zeta^{\alpha+1} &= Z\zeta^\alpha = \sqrt{2}HP^+\zeta^\alpha, \\ \chi^{\alpha+1} &= \bar{Z}\zeta^\alpha = \sqrt{2}HP^-\zeta^\alpha. \end{aligned} \tag{28}$$

Then ζ^α can be reconstructed using

$$\zeta^\alpha = Z^*\zeta^{\alpha+1} + \bar{Z}^*\chi^{\alpha+1}. \tag{29}$$

Like the real algorithm, this can be iterated. By proposition 9,

$$\begin{aligned} I &= \bar{Z}^*\bar{Z} + Z^*Z = \bar{Z}^*\bar{Z} + Z^*(\bar{Z}^*\bar{Z} + Z^*Z)Z = \dots \\ &= \sum_{\alpha=1}^N (Z^*)^{\alpha-1} \bar{Z}^*\bar{Z} Z^{\alpha-1} + (Z^*)^N Z^N. \end{aligned} \tag{30}$$

Hence for any $\zeta^0 \in V^c$ and $N \in \mathbb{N}$,

$$\begin{aligned}\zeta^0 &= \sum_{\alpha=1}^N (Z^*)^{\alpha-1} \bar{Z}^* \bar{Z} Z^{\alpha-1} \zeta^0 + (Z^*)^N Z^N \zeta^0 \\ &= \sum_{\alpha=1}^N (Z^*)^{\alpha-1} \bar{Z}^* \chi^\alpha + (Z^*)^N \zeta^N,\end{aligned}\tag{31}$$

where

$$\chi^\alpha \equiv \bar{Z} Z^{\alpha-1} \zeta^0, \quad \zeta^N \equiv Z^N \zeta^0.\tag{32}$$

The convergence and significance (in relation to the usual frequency decomposition) of this algorithm will be studied elsewhere. Here we merely note that the terms in the above sum are mutually orthogonal, since for $\beta > \alpha$ proposition 2.9 implies

$$\langle (Z^*)^{\alpha-1} \bar{Z}^* \chi^\alpha, (Z^*)^{\beta-1} \bar{Z}^* \chi^\beta \rangle = \langle \chi^\alpha, \bar{Z} (Z^*)^{\beta-\alpha} \bar{Z}^* \chi^\alpha \rangle = 0.\tag{33}$$

Just as the filters H and K have associated averaging- and “differencing” operators $h(S)$ and $g(S)$, there are operators associated with Z and \bar{Z} . For

$$\begin{aligned}Z^* \zeta(S) \phi &= 2^{-1/2} (h(S) \zeta(S^2) - ig(S) \zeta(S^{-2})) \phi \\ &= \left(\frac{h(S) - ig(S)C}{\sqrt{2}} \right) E^* \zeta(S) \phi,\end{aligned}\tag{34}$$

hence $Z^* = z^* E^*$ and, similarly, $\bar{Z}^* = \bar{z}^* E^*$, where

$$z^* = \frac{h(S) - ig(S)C}{\sqrt{2}}, \quad \bar{z}^* = \frac{h(S) + ig(S)C}{\sqrt{2}}.\tag{35}$$

Note, however, that these operators do not commute with S , since they contain C .

Proposition 10. *The operators z and \bar{z} satisfy*

$$zz^* = z^*z = \bar{z}\bar{z}^* = \bar{z}^*\bar{z} = I.\tag{36}$$

Proof. A straight computation gives

$$\begin{aligned}
z^*z &= \frac{1}{2} (h(S) - ig(S)C) (h(S^{-1}) + iCg(S^{-1})) \\
&= \frac{1}{2} (h(S)h(S^{-1}) + g(S)g(S^{-1})) + \frac{i}{2} (h(S)g(S) - g(S)h(S)) C \\
&= \frac{1}{2} (h(S)h(S^{-1}) + h(-S)h(-S^{-1})) \\
&= DEh(S)h(S^{-1})E^*D^{-1} = I
\end{aligned} \tag{37}$$

by proposition 2.5, and the other identities are proved similarly. \blacksquare

Furthermore, there is a natural complex function which stands in the same relation to z as do ϕ and ψ stand to $h(S)$ and $g(S)$, respectively. Consider the function $\bar{\chi} \in V_{-1}^+$ defined by

$$D\bar{\chi} \equiv Z^*\phi = z^*\phi = \frac{h(S) - ig(S)}{\sqrt{2}}\phi. \tag{38}$$

using $D\phi = h(S)\phi$ and $D\psi = g(S)\phi$, we get

$$\bar{\chi} = \frac{\phi - i\psi}{\sqrt{2}}. \tag{39}$$

Similarly, we define $\chi \in V_{-1}^-$ by

$$\chi \equiv D^{-1}\bar{Z}^*\phi = \frac{\phi + i\psi}{\sqrt{2}}. \tag{40}$$

The functions ϕ and ψ are somewhat reminiscent of the cosine- and sine-function. By that analogy, χ and $\bar{\chi}$ correspond to *the complex exponentials!* It may be that χ and $\bar{\chi}$, which combine averaging and “differencing” in a complex way, play a similar role in wavelet analysis as do the complex exponentials in Fourier analysis.

Finally, note that the Hermitian inner product in V^c has the decomposition

$$\langle \zeta, \zeta' \rangle = \langle \zeta, P^+\zeta' \rangle + \langle \zeta, P^-\zeta' \rangle, \tag{41}$$

which is the sum of the inner products in V^+ and V^- . Now the restriction of P^+ to $V \subset V^c$ maps V one-to-one onto V^+ , although it cannot preserve the inner product since V is real while V^+ is complex. In fact, for $\zeta \equiv P^+u$ and $\zeta' \equiv P^+u'$ in V^+ , we have

$$\begin{aligned}
\langle \zeta, \zeta' \rangle &= \langle u, P^+ u' \rangle = \frac{1}{2} \langle u, u' \rangle - \frac{i}{2} \langle u, Ju' \rangle \\
&\equiv \frac{1}{2} \langle u, u' \rangle - i\omega(u, u'),
\end{aligned}
\tag{42}$$

which states that imaginary part of the Hermitian inner product in V^+ is the skew-symmetric bilinear form ω . This form is known as the *symplectic structure* determined by the complex structure J together with the inner product on V . It plays a fundamental role in a broad variety of subjects, e.g. group representation theory, classical mechanics and quantum mechanics (see Marsden [1981]). I do not know whether it has any significance for wavelet analysis, but that seems to me a question worth exploring.

Some final remarks.

1. Although no attempt has been made here to work in an *abstract* setting, the algebraic approach clearly lends itself to such generalizations. We have made use of just a few facts about our initial set-up, for example that the inner product is invariant under S and D and also under C and M . This suggests that one *begin* with an algebra whose generators include S and D and other operators such as C and M , subject to certain relations, and look for *representations* of this algebra, i.e. for a vector space on which the elements of the algebra act as operators. In our case, the vector ϕ provides a representation on $L^2(\mathbb{R})$. ϕ is a *cyclic* vector because its orbit under the algebra spans $L^2(\mathbb{R})$, and the representation is *irreducible*. The representation is also *orthogonal* in the sense that the generators are represented by orthogonal operators. Solving the dilation equation $D\phi = h(S)\phi$ therefore amounts to constructing the entire representation!

2. The complex combinations $H \pm iK$ are reminiscent of certain operators that occur in quantum mechanics in connection with *coherent states* and which are also associated with time-frequency localization. See Kaiser [1990].

I thank Yuli Makovoz for helpful discussions.

4. Appendix

Here we assemble various information for the reader's convenience. We begin with a summary of the operational calculus.

1. Operational Calculus and Basis Representations

We list the actions of various operators in their “home versions”, both intrinsically and on the orthonormal bases $\{\phi_n\}$ of $V \equiv V_0$ and $\{\psi_n\}$ of $W \equiv W_0$. (See Daubechies [1988b].) Since $\varepsilon(S)$ is odd, it may be written in the form $\varepsilon(S) = \sum_k \varepsilon_k S^{2k+1}$.

$$\begin{aligned}
 H^*u(S)\phi &= h(S)u(S^2)\phi & H^*\phi_k &= \sum_n h_{n-2k}\phi_n \\
 Hu(S)\phi &= Eh(S^{-1})u(S)\phi & H\phi_n &= \sum_k h_{n-2k}\phi_k \\
 K^*u(S)\phi &= g(S)u(S^{-2})\phi & K^*\phi_k &= \sum_n g_{n+2k}\phi_n \\
 Ku(S)\phi &= Eg(S)u(S^{-1})\phi & K\phi_n &= \sum_k g_{n+2k}\phi_k \\
 G^*u(S)\psi &= g(S)u(S^2)\phi & G^*\psi_k &= \sum_n g_{n-2k}\phi_n \\
 Gu(S)\phi &= Eg(S^{-1})u(S)\psi & G\phi_n &= \sum_k g_{n-2k}\psi_k \\
 Ju(S)\phi &= \varepsilon(S)u(-S^{-1})\phi & J\phi_n &= (-1)^n \sum_k \varepsilon_k \phi_{2k+1-n}.
 \end{aligned}$$

2. Naturality

Naturality relates the scale-preserving filters $H_\alpha, K_\alpha, G_\alpha$ and Z_α to one another and to their home versions, which involve changes of scale. We give the relations for Z_α ; the others are similar. The last two equations show the advantage of the home versions for iterated decomposition and reconstruction.

$$\begin{aligned}
 Z_\alpha D^\alpha &= D^\alpha Z_0 = D^{\alpha+1} Z \\
 Z_\alpha^* D^{\alpha+1} &= D^\alpha Z_0^* D = D^\alpha Z^* \\
 Z_{\alpha+N} Z_{\alpha+N-1} \cdots Z_\alpha D^\alpha &= D^{\alpha+N+1} Z^{\alpha+N+1} \\
 Z_\alpha^* Z_{\alpha+1}^* \cdots Z_{\alpha+N}^* D^{\alpha+N+1} &= D^\alpha (Z^*)^{N+1}.
 \end{aligned}$$

3. Relation to Daubechies' Notation

Upon taking the Fourier transform, our operators $u(S)$ become functions $u(e^{i\xi T})$ on the unit circle. Their connections with those occurring in Daubechies' paper, where the sampling interval $T = 1$, are as follows: Write $\varepsilon(S) = S\varepsilon_-(S^2)$. Then

$$\begin{aligned}
 h(e^{i\xi}) &= H(\xi), & g(e^{i\xi}) &= G(\xi) \\
 h_+(e^{i\xi}) &= \alpha(\xi), & g_+(e^{i\xi}) &= \gamma(\xi) \\
 h_-(e^{i\xi}) &= \beta(\xi), & g_-(e^{i\xi}) &= \delta(\xi) \\
 \varepsilon_-(e^{i\xi}) &= -e^{i\lambda(\xi)}, & U(e^{i\xi}) &= M(\xi).
 \end{aligned}$$

4. Analogy with Complex Numbers

We give a “pocket dictionary” of the correspondence between wavelet operations and operations on complex numbers. This is done for the relation $V_0 \approx V_1 \oplus iV_1$, although the same analogy holds at every scale.

$$\begin{array}{rcl}
 V_1 & \longleftrightarrow & \mathbb{R} \\
 W_1 & \longleftrightarrow & i\mathbb{R} \\
 V_0 = V_1 \oplus JV_1 & \longleftrightarrow & \mathbb{C} = \mathbb{R} \oplus i\mathbb{R} \\
 u(S)\phi \mapsto H_0 u(S)\phi & \longleftrightarrow & x + iy \mapsto \Re(x + iy) \\
 u(S)\phi \mapsto K_0 u(S)\phi & \longleftrightarrow & x + iy \mapsto \Im(x + iy) \\
 u(S)\phi \mapsto Ju(S)\phi & \longleftrightarrow & z \mapsto iz \\
 Dv(S)\phi \mapsto H_0^* Dv(S)\phi & \longleftrightarrow & x \mapsto x + i0 \\
 Dw(S)\phi \mapsto K_0^* Dw(S)\phi & \longleftrightarrow & y \mapsto 0 + iy \\
 u(S)\phi = H_0^* Dv(S)\phi + JH_0^* Dw(S)\phi & \longleftrightarrow & z = x + iy \\
 Dw(S)\phi \mapsto R_1^* Dw(S)\phi & \longleftrightarrow & y \in \mathbb{R} \mapsto iy \in i\mathbb{R}.
 \end{array}$$

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