

# LOCAL FOURIER ANALYSIS AND SYNTHESIS

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## ABSTRACT

Fourier analysis and synthesis localized by the use of windows is developed. The analysis is performed by using a window called "analyzer," and the synthesis uses a dual window called "synthesizer." A sampling theorem is given in which the values of the localized Fourier transform on a sufficiently fine lattice are used to reconstruct the original signal. Various applications are proposed.

This paper was the text of a research proposal (MCS-7822673) submitted to the U.S. National Science Foundation in 1978. The proposal was not funded, but since the subject has attracted some interest recently, I felt it would be useful to make it available as a preprint.

## 1. Introduction

The object of the research proposed here is to develop and apply Local Fourier Analysis-Synthesis, a new mathematical method to be described below. This method sprang out of the recently developed phase-space formulation of relativistic quantum mechanics [1-4]. (See the reprints in the Appendix).

## 2. Local Fourier Analysis-Synthesis

The idea is best explained by considering a familiar example. Let  $f(x)$  represent a radiating electromagnetic field in  $n$ -dimensional space-time (we will assume, for simplicity, that  $f$  is real-valued rather than tensor-valued.) The usual method for extracting color-information from  $f$  is to consider its Fourier transform  $\hat{f}(k)$ . However, what we experience is color bounded by form. This is clearly not described by  $\hat{f}$  (which only tells us about color) nor by  $f$  (which only tells us about form). We seek, therefore, a function  $\tilde{f}(x,k)$  which gives the spectrum in the locality of of the space-time point  $x$ . Such a function can be obtained as follows: Let  $g(x)$  be a fixed function vanishing outside of a small neighborhood of the origin (this restriction turns out to be unnecessary.) Then the function  $f_y(x) = g(x-y)f(x)$  vanishes outside of a small neighborhood of  $y$  and will be thought of as the function  $f(x)$  localized about the point  $y$ .

Now define

$$\begin{aligned} \tilde{f}(y, k) &= \int_{R^n} e^{2\pi i k \cdot x} g(x-y) f(x) d^n x \\ (1) \quad &= \hat{f}_y(k) \end{aligned}$$

(we denote the usual Fourier transform by  $\hat{\phantom{x}}$  and its inverse by  $\check{\phantom{x}}$ .) We will refer to  $\tilde{f}$  as the local Fourier transform of  $f$ . The function  $g$  will be called an analyzer. It is the choice of  $g$  that determines what is to be considered microscopic and what macroscopic. If we are concerned with vision, for example, the spatial support of  $g$  would be the size of the pupil and its temporal form a reasonable impulse response function for the eye. To recover  $f$  from  $\tilde{f}$ , choose another fixed function  $h(x)$  with the property that

$$(2) \quad \int_{R^n} h(x) g(x) d^n x = 1.$$

Then it follows that

$$(3) \quad f(x) = \int_{R^{2n}} e^{-2\pi i k \cdot x} h(x-y) \tilde{f}(y, k) d^n y d^n k.$$

The function  $h$  will be called a synthesizer. Of course, there is a great deal of freedom involved in the choice of  $g$  and  $h$ , which can be used to simulate a variety of real situations.

It can be shown that the function  $\tilde{f}(y, k)$  is much smoother in both variables than either  $f(x)$  or  $\hat{f}(k)$ , hence lends itself very well to sampling, or computer analysis. In fact,

we have the following

Sampling Theorem. Let  $f \in L^2(\mathbb{R}^n)$ , let  $\sigma$  and  $\mu$  be positive numbers with  $\sigma \mu < 1$  and suppose that  $g(x)$  is such that the function  $\phi(x) = \mu^n \left[ \sum_{l \in \Lambda} |g(x-l\sigma)|^2 \right]^{-1}$

is bounded everywhere, where the sum extends over the unit lattice  $\Lambda = \{l = (l_1, \dots, l_n) \mid l_i \in \mathbb{Z}\}$ .

Let

$$f_{1m} = \tilde{f}(l\sigma, m\mu), \quad l, m \in \Lambda.$$

Then  $f(x)$  can be recovered (in the sense of  $L^2(\mathbb{R}^n)$ ) from the  $f_{1m}$ 's by

$$(4) \quad f(x) = \phi(x) \sum_{l, m \in \Lambda} e^{-2\pi i m \cdot x \mu} \overline{g(x-l\sigma)} f_{1m}.$$

The relation to quantum mechanics is as follows:  $f(x)$  corresponds to the coordinate representation,  $\hat{f}(k)$  to the momentum representation and  $\tilde{f}(y, k)$  to the phase-space representation. [1-8]. The sampling theorem is related to the uncertainty principle, which can be interpreted as stating that  $\tilde{f}$  cannot change drastically in a phase-space volume less than that of a Planck cell (this is essentially the condition  $\mu\sigma < 1$ ).

### 3. Proposed Applications

One possible application is to the problem used to motivate the above discussion, namely vision. That is, one might consider the proposition that the eye performs a kind of local Fourier analysis on the incoming light. As mentioned above, this is not unreasonable: the function  $g(x)$  is provided by the pupil aperture and the impulse response of the eye. If the observer's eye is located at  $y$ , he "sees" not the entire field  $f(x)$  but the localized version  $g(x-y)f(x)$ . The lens in his eye then Fourier transforms this function [9,10] onto the focal plane. Thus it seems that local Fourier analysis may provide a simple and natural model for vision.

More generally, it seems promising to use local Fourier analysis as a basis for the quantum theory of measurement [11]. The Bargmann-Segal, or "Coherent-State", representation [5-8] of quantum mechanics can be interpreted as a particular local-Fourier representation where  $g(x)$  has the form  $A \exp(-x^2/\sigma^2)$ . It was the realization that this restriction is unnecessary which led to the relativistic coherent-state representation [1-4]. To develop a theory of measurement, one might proceed as follows: Let  $R^4$  denote space-time and choose  $g(x)$  to have support in the backward open light cone

$$V_- = \left\{ x = (\vec{x}, x_0) \in R^4 \mid x_0 < -|\vec{x}| \right\} .$$

We interpret  $g$  as representing a measuring instrument which gives weight  $g(x)$  to an event at  $x$  when measurement occurs at the origin. Causality thus demands that  $g$  vanish outside of  $V_-$ . The local transform  $\tilde{f}(y,k)$  of a given field  $f(x)$  then represents the information gathered by the instrument  $g$  at the space-time point  $y$ . Equation (3) states that the entire field  $f$  can be recovered from the data gathered at all possible points  $y$ . (Of course, if  $f$  satisfies some hyperbolic differential equation, we can expect to recover it from data gathered on a single Cauchy surface.) Furthermore, the sampling theorem states that  $f$  can be recovered from data gathered at a discrete set of events, provided the net is sufficiently fine.

Other applications may suggest themselves from the theories of signal processing and linear systems [12,13].

Finally, note that local analysis-synthesis, as well as the sampling theorem, can be naturally generalized to the category of locally compact abelian groups [14]. This may further extend the range of possible applications.

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The author's proposal is a rather superficial description of a well-known area of research rather than a precise account of what he wants to do within that area. Moreover I find it difficult to understand from the vita why he has so many degrees and so few publications. On those grounds, I would rate his proposal only as FAIR, but looking through the research papers provided it seems that he has done some interesting work recently and that your rating GOOD may be more appropriate. I advise you to ask an expert on the Kostant-Souriau quantisation theory about the importance of his recent research.

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