Phase-space approach to relativistic quantum mechanics. I. Coherent-state representation for massive scalar particles

Gerald Kaiser

Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1
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We construct a family of equivalent representations U_λ ($\lambda > 0$) of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ for a massive scalar particle on spaces $\mathcal{K}_\lambda$ of functions defined over "phase space" P_λ. Each P_λ is a submanifold of the forward tube, and $\mathcal{K}_\lambda$ consists of restrictions on holomorphic solutions of the Klein–Gordon equation to P_λ. Each $\mathcal{K}_\lambda$ has a resolution of the identity in terms of "coherent states" e_z, $z \in P_\lambda$, which are wavepackets characterized by an invariant extremal property.

1. INTRODUCTION

This is the first in a series of papers devoted to a phase-space formulation of relativistic quantum mechanics. In this paper we construct representations of the "coherent-state" type for a free massive scalar particle. In forthcoming papers we extend the present formalism to particles with spin, supply our "phase spaces" with natural symplectic structures, and formulate a covariant phase-space quantization. The results of this paper were announced in Ref. 1.

We begin by sketching the coherent-state representation.

In addition to the well-known configuration-space and momentum-space representations of quantum mechanics for a nonrelativistic particle, there is a class of representations on spaces of functions over classical phase space,[2–7] the most common of which is known as the "coherent-state" representation. The simplest such representation[8] is constructed as follows: let X_k and P_k be the position and momentum operators for a particle in R^n ($k=1,\ldots,n$) and form the nonnormal operators $a_k = X_k + iP_k$. These are found to have an overcomplete set of eigenvectors $e_z : a_k e_z = \bar{z}_k e_z$ (the bar denotes complex conjugation), one for each $z = x - iy \in C^n$, and each e_z is a minimum-uncertainty wave packet with $\langle X_k \rangle = x_k$ and $\langle P_k \rangle = y_k$. The coherent-state representation is then the representation of wavefunctions f by functions $f(z) \equiv \langle e_z | f \rangle$. These functions are entire and satisfy

$$\langle f | g \rangle = \pi^{-n} \int_{C^n} \overline{f(z)} g(z) \exp(-|z|^2/2)\, d^{2n}z, \qquad (1.1)$$

where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, $d^{2n}z$ is Lebesgue measure, and the left-hand side denotes the inner product of f and g in the given Hilbert space $\mathcal{H}$ (say, of functions over configuration space).

In spite of its usefulness and intuitive appeal, the coherent-state representation is generally regarded as something of a fluke. The formal combinations $X_k \pm iP_k$, on which it is based, cannot be justified in physical terms, and the use of non-Hermitian operators as anything other than a technical device is regarded with suspicion.

It is one of the aims of this paper to show that representations similar to the above can in fact spring from physical principles, and that the resulting formalism can, as above, be interpreted as a phase-space representation of the given quantum system. The general argument goes as follows: The positivity of the quantum Hamiltonian permits the extension of the one-parameter unitary group $\exp(-itH)$ (t real) representing dynamics in $\mathcal{H}$ to a holomorphic semigroup $\exp(-i\tau H)$ ($\tau = t - i\beta$, $\beta > 0$). On a classical level, evolution in complex time (were it possible) would result in a complexification of the configuration space (hence complex space-time). This has a counterpart at the quantum level in that wavefunctions evolved in complex time, $\exp(-i\tau H)f = \exp(-itH)\exp(-\beta H)f$, may be continued analytically from R^n (configuration space) to a subset (possible all) of C^n. In particular, if the given system is a free nonrelativistic particle, this continuation is even possible at the classical level and gives the complexified position $\mathbf{z}(\tau) = \mathbf{x}_0 + \tau(\mathbf{p}/m) = (\mathbf{x}_0 + t\mathbf{p}/m) - i\beta\mathbf{p}/m$, which is a combination of the type $\mathbf{x} - i\mathbf{p}$. Hence the complexified space can, at every complex "instant" $t - i\beta$, be interpreted as a classical phase space. Moreover, the set of analytically continued solutions carries a representation of the quantum dynamics on functions over phase space.

In Sec. 2 we develop this idea for a free scalar nonrelativistic particle and arrive at a representation which essentially coincides with the usual coherent-state representation. An analogous construction is carried out in Sec. 3 for a relativistic free scalar particle (with positive mass). The ensuing formalism appears to be new and has the general features of the coherent-state representation. The "phase spaces" P_λ of Sec. 3 are products of R^n (configuration space) with an n-dimensional hyperboloid (roughly, a mass shell). It is shown that in the nonrelativistic limit ($c \to \infty$) the formalism goes over smoothly to the formalism of Sec. 2. In Sec. 4 we study the relativistic coherent states e_z, $z \in P_\lambda \approx C^n$. We show that e_{x-iy} is a wavepacket with $\langle X_k \rangle = x_k$ and $\langle P_k \rangle = b_\lambda v_k$, where b_λ is a constant and X_k are the position operators obtained by Newton and Wigner[11] by axiomatizing the notion of "localized states". These results partly justify calling P_λ a "phase space." The e_z are shown to be characterized by an extremal property which, we suggest, is a covariant substitute for minimal uncertainty.

2. NONRELATIVISTIC PARTICLE

The wave function of a free, spinless nonrelativistic particle in R^n evolves under the Schrödinger equation

$$i\frac{\partial f}{\partial t} = Hf, \quad H = -\frac{1}{2m}\Delta. \qquad (2.1)$$

The solutions are given by

$$f(x, t) = [\exp(-itH)f](\mathbf{x})$$
$$= (2\pi)^{-n/2} \int_{R^n} \exp(-itp^2/2m + i\mathbf{x}\cdot\mathbf{p})\hat{f}(\mathbf{p})\, d^n p, \qquad (2.2)$$

where $\hat{f}(\mathbf{p})$ is the Fourier transform of the initial function $f(\mathbf{x}, 0) \in L^2(R^n)$. Now let $\mathbf{z} = \mathbf{x} - i\mathbf{y} \in C^n$ and let $\tau = t - i\beta$ be in the lower half-plane C^- $(\beta > 0)$. Then $\exp(-i\tau p^2/2m + i\mathbf{z}\cdot\mathbf{p})$ decays rapidly as $|\mathbf{p}| \to \infty$, and Eq. (2.2) defines a function $f(\mathbf{z}, \tau) \equiv [\exp(-i\tau H)f](\mathbf{z})$, holomorphic in $\mathcal{D} = C^n \times C^-$. Let $\mathcal{H} = \{f(\mathbf{z}, \tau) \mid \hat{f}(\mathbf{p}) \in L^2(R^n)\}$ be the vector space of all such functions. Then, for each $\beta > 0$, the function $f_\beta(\mathbf{z}) = f(\mathbf{z}, -i\beta) = [\exp(-\beta H)f](\mathbf{z})$ is entire in C^n. Let $\mathcal{H}_\beta$ be the space of all such functions $f_\beta(\mathbf{z})$. On $\mathcal{H}_\beta$ define the map $\exp(-itH)$ by

$$\exp(-itH)[\exp(-\beta H)f] = \exp(-\beta H)[\exp(-itH)f],$$
$$f \in L^2(R^n). \qquad (2.3)$$

We shall make $\mathcal{H}_\beta$ into a Hilbert space such that $t \to \exp(-itH)$ is a unitary representation of dynamics on $\mathcal{H}_\beta$.

Thus, let $\beta > 0$ and $\mathbf{z} = \mathbf{x} - i\mathbf{y} \in C^n$. Then

$$f_\beta(\mathbf{z}) = [\exp(-\beta H)f](\mathbf{z})$$
$$= (2\pi)^{-n/2} \int_{R^n} \exp(-\beta p^2/2m + i\mathbf{z}\cdot\mathbf{p})\hat{f}(\mathbf{p})\, d^n p$$
$$\equiv \langle e_{\mathbf{z}}^\beta | f \rangle, \qquad (2.4)$$

where

$$\langle e_{\mathbf{z}}^\beta | \mathbf{p} \rangle = (2\pi)^{-n/2} \exp(-\beta p^2/2m + i\mathbf{z}\cdot\mathbf{p}) \qquad (2.5)$$

with Fourier transform

$$\langle e_{\mathbf{z}}^\beta | \mathbf{x}' \rangle = (2\pi\beta/m)^{-n/2} \exp[-m(\mathbf{z} - \mathbf{x}')^2/2\beta]. \qquad (2.6)$$

The $e_{\mathbf{z}}^\beta$ are minimum-uncertainty spherical wavepackets with $\langle X_k \rangle = x_k$, $\langle P_k \rangle = (m/\beta)y_k$, $\Delta X_k = \sqrt{\beta/2m}$ and $\Delta P_k = \sqrt{m/2\beta}$. They are eigenvectors of $a_k(\beta) = X_k + i(\beta/m)P_k$ with eigenvalue $\bar{z}_k$.

For $f_\beta \in \mathcal{H}_\beta$ define

$$\|f\|_\beta^2 = \int_{C^n} |f_\beta(\mathbf{z})|^2\, d\mu_\beta(\mathbf{z}), \qquad (2.7)$$

where

$$d\mu_\beta(\mathbf{z}) = (m/\pi\beta)^{n/2} \exp(-my^2/\beta)\, d^n x\, d^n y. \qquad (2.8)$$

Theorem 1: Let $\beta > 0$ and $\hat{f}(\mathbf{p}) \in L^2(R^n)$. Then

$$\|f\|_\beta = \|\hat{f}\|. \qquad (2.9)$$

In particular,

(a) $\|\cdot\|_\beta$ is a norm on $\mathcal{H}_\beta$ under which $\mathcal{H}_\beta$ is a Hilbert space.

(b) The map $\exp(-itH)$ is unitary on $\mathcal{H}_\beta$.

(c) The map $\exp(-\beta H)$ is unitary from $L^2(R^n)$ onto $\mathcal{H}_\beta$ and intertwines the dynamics on $L^2(R^n)$ with the dynamics on $\mathcal{H}_\beta$.

Remark: Equation (2.9) can be polarized to give a resolution of the identity: For f, g, in $L^2(R^n)$,

$$\langle f | g \rangle_\beta \equiv \int_{C^n} \langle f | e_{\mathbf{z}}^\beta \rangle \langle e_{\mathbf{z}}^\beta | g \rangle\, d\mu_\beta(\mathbf{z})$$
$$= \langle f | g \rangle_{L^2(R^n)}. \qquad (2.10)$$

Hence $f \to \langle e_{\mathbf{z}}^\beta | f \rangle$ is a "representation" of f by an entire function. The connection with the coherent-state representation is as follows: Set $m = \beta = 1$ and let $\tilde{f}(\mathbf{z}) = \pi^{n/4} \times \exp(z^2/4) f_\beta(\mathbf{z})$. Then (2.10) becomes

$$\pi^{-n} \int_{C^n} \overline{\tilde{f}(\mathbf{z})}\tilde{g}(\mathbf{z}) \exp(-\tfrac{1}{2}|\mathbf{z}|^2)\, d^n x\, d^n y = \langle f | g \rangle_{L^2(R^n)}, \qquad (2.11)$$

so that $f \to \tilde{f}(\mathbf{z})$ is (essentially) the ordinary coherent-state representation [in most of the literature, $\mathbf{z} = (\mathbf{x} - i\mathbf{y})/\sqrt{2}$; the weight function is then $\exp(-|\mathbf{z}|^2)$].

Proof: Let $\hat{f} \in \mathcal{S}(R^n)$. By (2.4), $f_\beta(\mathbf{x} - i\mathbf{y}) = \check{g}_{\beta,\mathbf{y}}(\mathbf{x})$ where $g_{\beta,\mathbf{y}}(\mathbf{p}) = \exp(-\beta p^2/2m + \mathbf{y}\cdot\mathbf{p})\hat{f}(\mathbf{p})$ and $\check{g}$ denotes the inverse Fourier transform of g. Thus, by Plancherel's theorem (and Fubini's),

$$\|f\|_\beta^2 = (m/\pi\beta)^{n/2} \int \exp(-my^2/\beta)\, dy$$
$$\times \int \exp(-\beta p^2/m + 2\mathbf{y}\cdot\mathbf{p}) |\hat{f}(\mathbf{p})|^2\, dp$$
$$= \int |\hat{f}(\mathbf{p})|^2\, dp = \|\hat{f}\|^2,$$

which proves (2.9) for $\hat{f} \in \mathcal{S}(R^n)$, hence also for $\hat{f} \in L^2(R^n)$ by continuity. (a)–(c) are obvious. ■

For the definition of intertwining operators, see Ref. 12.

3. RELATIVISTIC PARTICLE

In the last section we obtained unitary maps from $L^2(R^n)$ onto Hilbert spaces $\mathcal{H}_\beta$ where the role of δ functions is played by spherical wavepackets $e_{\mathbf{z}}^\beta$ in $L^2(R^n)$ [$\mathcal{H}_\beta$ is continuously imbedded in $L^2(R^n)$ by restricting $f_\beta(\mathbf{z})$ to R^n]. This formalism is nonrelativistic since the inner product for $L^2(R^n)$ is not Lorentz-invariant. In this section we define covariant counterparts of the $e_{\mathbf{z}}^\beta$ and prove the analog of Theorem 1. We begin with the relativistic version of the free-particle Schrödinger equation, namely the Klein–Gordon equation (for a free scalar particle of mass $m > 0$ in $n+1$ space–time dimensions):

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta - m^2\right) f(\mathbf{x}, t) = 0. \qquad (3.1)$$

The positivity of the energy played an essential role in Sec. 2, and will do so again here. Hence we confine ourselves to positive-energy solutions.[13] These are given by

$$f(x) = f(\mathbf{x}, x_0) = [\exp(-ix_0 H)f](\mathbf{x})$$
$$= (2\pi)^{-n/2} \int_{R^n} \exp(-ixp)\hat{f}(\mathbf{p})\, d\Omega(\mathbf{p}), \qquad (3.2)$$

where

$$x_0 = ct, \quad H = +(m^2c^2 - \Delta)^{1/2}, \quad xp = x_0\omega - \mathbf{x}\cdot\mathbf{p},$$
$$\omega = (m^2c^2 + \mathbf{p}^2)^{1/2}, \quad d\Omega(\mathbf{p}) = d^n p/\omega$$

is the Lorentz-invariant measure on the mass shell,[15] and $\hat{f}(\mathbf{p})/\omega$ is the ordinary Fourier transform of the initial function $f(\mathbf{x}, 0)$ (considered, say, as a tempered distribution in R^n). For every $\hat{f}(\mathbf{p}) \in L^2(\Omega)$, i.e., with

$$\|\hat{f}\|^2 \equiv \int_{R^n} |\hat{f}(\mathbf{p})|^2\, d\Omega(\mathbf{p}) < \infty, \qquad (3.3)$$

the corresponding solution $f(x)$ is the boundary value of a function $f(z)$ holomorphic in the *forward tube*[14,15]

$$\mathcal{T}=R^{n+1}-iV_+=\{z=x-iy\,|\,x\in R^{n+1},\ y\in V_+\},$$

where $V_+=\{y=(y_0,\mathbf{y})\in R^{n+1}\,|\,y_0>|\mathbf{y}|\}$ is the open forward light cone in R^{n+1}. This is because $|\exp(-izp)|$ $\equiv|\exp(-iz_0\omega+i\mathbf{z}\cdot\mathbf{p})|=\exp(-y_0\omega+\mathbf{y}\cdot\mathbf{p})\equiv\exp(-yp)$ $<\exp[-(y_0-|\mathbf{y}|)|\mathbf{p}|]$; hence $|\exp(-izp)|$ decays rapidly as $|\mathbf{p}|\to\infty$ for fixed $z=x-iy\in\mathcal{T}$. $\mathcal{T}$ will replace the domain $\mathcal{D}=C^n\times C^-$ of Sec. 2, and is strictly contained in $\mathcal{D}$. Thus, for $z\in\mathcal{T}$,

$$f(z)=(2\pi)^{-n/2}\int_{R^n}\exp(-izp)\hat{f}(\mathbf{p})\,d\Omega(\mathbf{p})$$
$$\equiv\langle e_z|f\rangle,\tag{3.4}$$

where

$$\langle e_z|p\rangle=(2\pi)^{-n/2}\exp(-izp)\tag{3.5}$$

and $\langle e_z|f\rangle$ denotes the inner product in $L^2(\Omega)$. The vectors e_z are in $L^2(\Omega)$, since for $z,w\in\mathcal{T}$,

$$\langle e_z|e_w\rangle=(2\pi)^{-n}\int_{R^n}\exp[-i(z-\bar{w})p]\,d\Omega(\mathbf{p})$$
$$=(2/i)\Delta_+(z-\bar{w})$$
$$=(1/\pi)(mc/2\pi\eta)^{\nu}K_\nu(\eta mc),\tag{3.6}$$

where Δ_+ is the familiar two-point function for the free scalar field of mass m,[14] $\eta=[-(z-\bar{w})^2]^{1/2}\equiv[-(z_0-\bar{w}_0)^2+\sum_1^n(z_k-\bar{w}_k)^2]^{1/2}$ is uniquely defined by analytic continuation from $\eta=[-(z-\bar{z})^2]^{1/2}=2(y_0^2-\mathbf{y}^2)^{1/2}$ when $z=w=x-iy\in\mathcal{T}$, K_ν is a modified Bessel function,[16] and throughout the rest of this paper we set $\nu=(n-1)/2$. The analog of the space $\mathcal{H}$ of holomorphic solutions $f(\mathbf{z},\tau)$ in $\mathcal{D}$ is the space $\mathcal{K}=\{f(z)\,|\,f\in L^2(\Omega)\}$ of functions defined by (3.4). Recall now that $\mathcal{H}_\beta$ was obtained from $\mathcal{H}$ by restricting functions $f\in\mathcal{H}$ to the "phase space" $P_\beta^{\rm NR}=\{(\mathbf{z},-i\beta)\,|\,\mathbf{z}\in C^n\}\approx C^n$. This set is, however, not contained in $\mathcal{T}$. To obtain a relativistic phase space we reason as follows: $\mathcal{D}$ can be obtained (roughly) as a deformation of $\mathcal{T}$ by letting $c\to\infty$ while fixing $\tau=z_0/c$. A set in $\mathcal{T}$ which goes into $P_\beta^{\rm NR}$ under this "deformation" is

$$P_\lambda=\{(\mathbf{z},-i(\lambda^2+\mathbf{y}^2)^{1/2})\,|\,\mathbf{z}=\mathbf{x}-i\mathbf{y}\in C^n\},\tag{3.7}$$

with $\lambda=\beta c>0$. We will show that P_λ is a suitable phase space. For $\lambda=0$ Eq. (3.7) defines P_0 as a subset of the boundary of $\mathcal{T}$.

The sets P_λ are clearly not invariant under Lorentz transformations. To make the formalism manifestly Poincaré-covariant, we will also need the sets

$$P'_\lambda=\{(\mathbf{z},x_0-i(\lambda^2+\mathbf{y}^2)^{1/2})\,|\,\mathbf{z}=\mathbf{x}-i\mathbf{y}\in C^n,\quad x_0\in R\}.\tag{3.8}$$

Every function $f(z)\in\mathcal{K}$ defines a "boundary value" function on P'_0 (and by restriction also on P_0) as follows: for $z=(\mathbf{x}-i\mathbf{y},x_0-i|\mathbf{y}|)\in P'_0$,

$$f(z)\equiv f_0(\mathbf{x}-i\mathbf{y},x_0)$$
$$=((1/\omega)\exp(-|\mathbf{y}|\omega+\mathbf{y}\cdot\mathbf{p}-ix_0\omega)\hat{f}(\mathbf{p}))^{\check{}}(\mathbf{x})\tag{3.9}$$

[see (3.4)]. It follows from (3.9) that, for fixed $\mathbf{y}\in R^n$ and $x_0\in R$, f_0 is in $L^2(R^n)$ as a function of $\mathbf{x}$. [Actually, as we shall see, $f_0\in L^2(C^n)$ in $\mathbf{x}-i\mathbf{y}$ and $f\to f_0$ in $L^2(C^n)$ as $\lambda\to 0$.] Thus $f(z)$ makes sense even when $z\in P'_0$ (though its pointwise values no longer have meaning).

Given $\lambda\geqslant 0$ and $f\in\mathcal{K}$, define

$$\|f\|_\lambda^2=\int_{P_\lambda}|f(z)|^2\,d\mu_\lambda(z),\tag{3.10}$$

where

$$d\mu_\lambda(z)=C_\lambda\,d^n\mathbf{x}\,d^n\mathbf{y},\quad z=(\mathbf{x}-i\mathbf{y},z_0)\in P_\lambda\tag{3.11}$$

with

$$C_\lambda=[(2/\pi)(\pi\lambda/mc)^{\nu+1}K_{\nu+1}(2\lambda mc)]^{-1},\quad\lambda>0,\tag{3.12}$$

and $C_0\equiv\lim_{\lambda\to 0}C_\lambda=(mc)^{n+1}/\pi^\nu\Gamma(\nu+1)$. C_λ is a continuous, monotone increasing function of λ on $[0,\infty)$, with

$$C_\lambda\sim mc(mc/\pi\lambda)^{n/2}\exp(2\lambda mc)\quad\text{as}\quad\lambda mc\to\infty.\tag{3.13}$$

These facts, and others needed later, follow from certain properties of the K_ν,[16] which we summarize in Appendix A. We may regard $d\mu_\lambda$ either as a measure on P_λ or as a measure on C^n. In the latter interpretation (which will also be useful) we write (3.10) as

$$\|f\|_\lambda^2=\int_{C^n}|f_\lambda(\mathbf{z})|^2\,d\mu_\lambda(\mathbf{z}),\tag{3.14}$$

where $f_\lambda(\mathbf{z})=f(\mathbf{z},-i(\lambda^2+\mathbf{y}^2)^{1/2})$ is the restriction of $f\in\mathcal{K}$ to P_λ. Let $\mathcal{K}_\lambda=\{f_\lambda(\mathbf{z})\,|\,f\in\mathcal{K}\}$ be the space of all such restrictions (boundary values, if $\lambda=0$) and denote the map $\hat{f}(\mathbf{p})\to f_\lambda(\mathbf{z})$ from $L^2(\Omega)$ onto $\mathcal{K}_\lambda$ by D_λ. Similarly let $\mathcal{K}'_\lambda$ be the space of restrictions $f_\lambda(x,\mathbf{y})=f(\mathbf{x}-i\mathbf{y},\ x_0-i(\lambda^2+\mathbf{y}^2)^{1/2})$ to P'_λ and denote the corresponding map by D'_λ. Since each $f_\lambda(x,\mathbf{y})\in\mathcal{K}'_\lambda$ satisfies (3.1) in $x\in R^{n+1}$, $\mathcal{K}'_\lambda$ is simply the space of solutions with initial values in $\mathcal{K}_\lambda$. Notice that (3.14) is defined for $f_\lambda\in\mathcal{K}'_\lambda$ as well as for $f_\lambda\in\mathcal{K}_\lambda$.

Now $L^2(\Omega)$ carries a unitary, irreducible representation of the restricted Poincaré group $\mathcal{P}_+^\uparrow$,[14] given by

$$(U(a,\Lambda)\hat{f})(p)=\exp(iap)\hat{f}(\Lambda^{-1}p),\tag{3.15}$$

where $(a,\Lambda)\in\mathcal{P}_+^\uparrow$ acts on space–time according to

$$(a,\Lambda)x=\Lambda x+a,\quad x\in R^{n+1}.\tag{3.16}$$

In (3.15) $p=(\mathbf{p},\omega)$ denotes a point on the mass shell (a homogeneous space for the Lorentz group) rather than the corresponding momentum vector $\mathbf{p}$. The representation (3.15) defines a corresponding representation on $\mathcal{K}$ given by

$$(\tilde{U}(a,\Lambda)f)(z)=f(\Lambda^{-1}(z-a))\tag{3.17}$$

(where we have extended the action of $\mathcal{P}_+^\uparrow$ to $\mathcal{T}$ by linearity). Now P'_λ is a homogeneous space of $\mathcal{P}_+^\uparrow$ [in fact, $P'_\lambda\approx\mathcal{P}_+^\uparrow/\mathrm{SO}(n)$ since the stability subgroup at, say, $(0,-i\lambda)$ is $\mathrm{SO}(n)$]. Hence (3.17) gives a representation U'_λ on $\mathcal{K}'_\lambda$ by restriction (taking boundary values, if $\lambda=0$). Since extension sets up a one–one correspondence between $\mathcal{K}_\lambda$ and $\mathcal{K}'_\lambda$ we also have a representation U_λ on $\mathcal{K}_\lambda$, but this one is less direct since P_λ is not invariant under $\mathcal{P}_+^\uparrow$.

The next theorem, which is our first main result, shows that U, U_λ, and U'_λ are all unitarily equivalent.

Theorem 2: Let $\lambda\geqslant 0$ and $\hat{f}\in L^2(\Omega)$. Then

$$\|f\|_\lambda=\|\hat{f}\|.\tag{3.18}$$

In particular,

(a) $\|\cdot\|_\lambda$ is a norm on $\mathcal{K}_\lambda$ ($\mathcal{K}'_\lambda$) under which $\mathcal{K}_\lambda$ ($\mathcal{K}'_\lambda$) is Hilbert space.

(b) U_λ (U'_λ) is a unitary irreducible representation of $\mathcal{P}^{\uparrow}_{+}$ on $\mathcal{K}_\lambda$ ($\mathcal{K}'_\lambda$).

(c) D_λ (D'_λ) is unitary from $L^2(\Omega)$ onto $\mathcal{K}_\lambda$ ($\mathcal{K}'_\lambda$) and intertwines the representations U and U_λ (U'_λ) of $\mathcal{P}^{\uparrow}_{+}$.

Proof: The proof is completely parallel to that of Theorem 1. Let $\hat{f} \in \mathcal{S}(R^n)$ and note that

$$f(z) = (2\pi)^{-n/2} \int_{R^n} \exp(-ix_0\omega + i\mathbf{x}\cdot\mathbf{p} - yp)\hat{f}(\mathbf{p})\, d^n p/\omega$$
$$= ((1/\omega)\exp(-ix_0\omega - yp)\hat{f})\check{}\,(\mathbf{x}). \qquad (3.19)$$

Hence

$$\|f\|^2_\lambda = C_\lambda \int d^n y \int d^n x \,|f(\mathbf{x} - i\mathbf{y}, -i(\lambda^2 + \mathbf{y}^2)^{1/2})|^2$$
$$= C_\lambda \int d^n y \int d^n p \,|(1/\omega)\exp[-(\lambda^2+\mathbf{y}^2)^{1/2}\omega + \mathbf{y}\cdot\mathbf{p}] \times \hat{f}(\mathbf{p})|^2$$
$$= C_\lambda \int d^n p [|\hat{f}(\mathbf{p})|^2/\omega^2] \int d^n y \exp[-2(\lambda^2+\mathbf{y}^2)^{1/2}\omega + 2\mathbf{y}\cdot\mathbf{p}]$$
$$= \int (d^n p/\omega)|\hat{f}(\mathbf{p})|^2 = \|\hat{f}\|^2,$$

where we have used (A6) with $\alpha = 0$. This proves (3.18) for $\hat{f} \in \mathcal{S}(R^n)$, hence for $\hat{f} \in L^2(\Omega)$ by continuity. (a) and (b) are obvious, and the intertwining property follows from

$$(D'_\lambda U(a,\Lambda)\hat{f})(z) = (D'_\lambda(\exp(iap)\hat{f}(\Lambda^{-1}p)))(z)$$
$$= (2\pi)^{-n/2} \int \exp(-izp + iap)\hat{f}(\Lambda^{-1}p)\, d\Omega(p)$$
$$= (2\pi)^{-n/2} \int \exp[-i(z-a)\Lambda p'] \times \hat{f}(p')\, d\Omega(\Lambda p')$$
$$= (2\pi)^{-n/2} \int \exp[-i(\Lambda^{-1}(z-a))p'] \times \hat{f}(p')\, d\Omega(p')$$
$$= f(\Lambda^{-1}(z-a)) = (U'_\lambda D'_\lambda f)(z), \quad z \in P'_\lambda,$$

where we have used the invariance of $d\Omega(p)$. ■

The norm $\|\cdot\|_\lambda$ on $\mathcal{K}_\lambda$ and $\mathcal{K}'_\lambda$ defines an inner product $\langle\cdot|\cdot\rangle_\lambda$ on these spaces by polarization. As an immediate consequence of Theorem 2 we have

Corollary 1: Let $\lambda \geq 0$ and $\hat{f}, \hat{g} \in L^2(\Omega)$. Then

$$\langle f|g\rangle_\lambda \equiv \int_{P_\lambda} \langle f|e_z\rangle\langle e_z|g\rangle\, d\mu_\lambda(z)$$
$$= \langle f|g\rangle_{L^2(\Omega)}. \qquad (3.20)$$

In particular, taking $\hat{f} = e_w$ ($w \in \mathcal{T}$), we obtain

$$g(w) = \langle e_w|g\rangle = \int_{P_\lambda} \langle e_w|e_z\rangle\langle e_z|g\rangle\, d\mu_\lambda(z)$$
$$= \int_{P_\lambda} \langle e_w|e_z\rangle g(z)\, d\mu_\lambda(z). \qquad (3.21)$$

Equation (3.21), restricted to $w \in P_\lambda$, states that $\langle e_w|e_z\rangle$ is a (hence *the*) reproducing kernel[17] for $\mathcal{K}_\lambda$.

In the sequel we will sometimes identify the spaces $L^2(\Omega)$, $\mathcal{K}$, $\mathcal{K}_\lambda$, and $\mathcal{K}'_\lambda$ (as Theorem 2 permits us to do). Thus f could stand for $\hat{f}(\mathbf{p})$ or $f(z)$ as an element of $\mathcal{K}$, $\mathcal{K}_\lambda$, or $\mathcal{K}'_\lambda$. We will also set $c = 1$ except in considerations involving the nonrelativistic limit.

We can now make precise the sense in which a function $f \in \mathcal{K}$ takes on its boundary value on P_0.

Corollary 2: (a) Each $\mathcal{K}_\lambda$ is a closed subspace of $L^2(C^n)$. (b) Let $0 \leq \lambda < \lambda'$ and $\hat{f} \in L^2(\Omega)$. Then

$$\|f_{\lambda'} - f_\lambda\|_{L^2(C^n)} \to 0 \quad \text{as } \lambda' \downarrow \lambda.$$

Proof: (a) follows from (3.18), and (b) follows essentially from the proof of Theorem 2:

$$\|f_{\lambda'} - f_\lambda\|^2_{L^2(C^n)} = \int dp[|\hat{f}(\mathbf{p})|^2/\omega^2] \int dy \times (\exp[-(\lambda^2+\mathbf{y}^2)^{1/2}\omega] - \exp[-(\lambda'^2+\mathbf{y}^2)^{1/2}\omega])^2 \exp(2\mathbf{y}\cdot\mathbf{p})$$
$$= \int dp[|\hat{f}(\mathbf{p})|^2/\omega^2][\omega/C_\lambda + \omega/C_{\lambda'} - 2J(p,\lambda,\lambda')]$$

where

$$J = \int dy \exp\{-[(\lambda^2+\mathbf{y}^2)^{1/2} + (\lambda'^2+\mathbf{y}^2)^{1/2}]\omega + 2\mathbf{y}\cdot\mathbf{p}\}.$$

But $\omega/C_{\lambda'} \leq J \leq \omega/C_\lambda$; hence

$$\|f_{\lambda'} - f_\lambda\|^2_{L^2(C^n)} \leq (C_\lambda^{-1} - C_{\lambda'}^{-1})\|\hat{f}\|^2,$$

which implies (b). ■

We conclude this section by showing that the e_z-representation on $\mathcal{K}_\lambda$ is indeed a relativistic version of the e_z^β-representation on $\mathcal{H}_\beta$. For given $\beta > 0$, define

$$f_\beta^{\mathrm{NR}}(\mathbf{x} - i\mathbf{y}) = \exp(-m\mathbf{y}^2/2\beta)(\exp(-\beta\mathbf{p}^2/2m + \mathbf{y}\cdot\mathbf{p})\hat{f}(\mathbf{p}))\check{}\,(\mathbf{x})$$
$$= \left[\exp\left[-\frac{\beta m}{2}\left(\frac{\mathbf{p}}{m} - \frac{\mathbf{y}}{\beta}\right)^2\right]\hat{f}(\mathbf{p})\right]\check{}\,(\mathbf{x}). \qquad (3.22)$$

Theorem 3: Let $\beta > 0$ and $\hat{f}(\mathbf{p}) \in L^2(R^n)$. Then $f_\beta^{\mathrm{NR}}(\mathbf{z}) \in L^2(C^n)$ and

$$J(c) \equiv \|mc\exp(\beta mc^2)f_{\beta c} - f_\beta^{\mathrm{NR}}\|^2_{L^2(C^n)} \to 0 \quad \text{as } c \to \infty,$$

where $f_{\beta c}$ is the function in $\mathcal{K}_{\beta c}$ corresponding to $\hat{f} \in L^2(R^n) \subset L^2(\Omega)$. The proof is given as Appendix B.

4. THE WAVEPACKETS e_z

In this section we study the "relativistic coherent states" e_z. We show that they are centered about $\mathbf{x} = \mathrm{Re}(\mathbf{z})$, travel with average momentum proportional to $\mathbf{y}$, and are characterized by a property which is a covariant analog of minimal uncertainty.

To compute the position of the center of e_z, we need position operators. It was shown by Newton and Wigner[11] that certain group-theoretical postulates about (idealized) "localized states"—e.g., that any space translate of a localized state be "orthogonal" to the state[18]—uniquely determine a set of self-adjoint operators [here given on $L^2(\Omega)$]

$$X_k = i\left(\frac{\partial}{\partial p_k} - \frac{p_k}{2\omega^2}\right), \quad k = 2, \ldots, n, \qquad (4.1)$$

whose (generalized) eigenvectors are the localized states. (The notion of being localized in this sense, however, depends on the frame of reference.) In a later paper, dealing with quantization, we will show that the operators (4.1) can also be obtained naturally from the formalism of Sec. 3. For the purpose of this section, we simply adopt (4.1) as the definition of position operators.

We begin by computing the expectation of X_k in e_z:

$$\langle e_z|X_k e_z\rangle = \int \frac{dp}{\omega}\langle e_z|p\rangle i\left(\frac{\partial}{\partial p_k}-\frac{p_k}{2\omega^2}\right)\langle p|e_z\rangle$$

$$=\int dp\,\frac{\langle e_z|p\rangle}{\sqrt{\omega}}\,i\frac{\partial}{\partial p_k}\left(\frac{\langle p|e_z\rangle}{\sqrt{\omega}}\right)$$

$$=\mathrm{Re}\int dp\,\frac{\langle e_z|p\rangle}{\sqrt{\omega}}\,i\frac{\partial}{\partial p_k}\left(\omega^{-1/2}\frac{\exp(i\bar{z}p)}{(2\pi)^{n/2}}\right)$$

$$=x_k\langle e_z|e_z\rangle.$$

Thus

$$\langle X_k\rangle = x_k \equiv \mathrm{Re}(z_k). \tag{4.2}$$

To find the expectation of P_α, let

$$G(m,y)=\int_{R^n}\exp(-2yp)\,d\Omega(p)=2(\pi m/\lambda)^\nu K_\nu(2\lambda m)$$

$$=a(m)\varphi^{-\nu}K_\nu(\varphi)=b(y)\varphi^\nu K_\nu(\varphi), \tag{4.3}$$

where $a(m)=2(2m^2\pi)^\nu$, $b(y)=2(2\lambda^2/\pi)^{-\nu}$, $\varphi=2\lambda m$, and $\lambda=\lambda(y)\equiv(y_\alpha y^\alpha)^{1/2}$ with all the y_α considered as independent variables. $G(m,y)$ will be a "partition function" (as in statistical mechanics) for generating expectations. Thus, using (A2),

$$\int p_\alpha \exp(-2yp)\,d\Omega(p)=-\frac{1}{2}\frac{\partial G}{\partial y_\alpha}$$

$$=-2m^2y_\alpha a(m)\frac{1}{\varphi}\frac{\partial}{\partial\varphi}(\varphi^{-\nu}K_\nu(\varphi))$$

$$=2m^2y_\alpha a(m)\varphi^{-\nu-1}K_{\nu+1}(\varphi);$$

hence

$$\langle P_\alpha\rangle=-\frac{1}{2G}\frac{\partial G}{\partial y^\alpha}=\frac{K_{\nu+1}(2\lambda m)}{K_\nu(2\lambda m)}\cdot\frac{m}{\lambda}y_\alpha \tag{4.4}$$

in the state e_z $(z=x-iy)$. Similarly,

$$\int p_\alpha p_\beta\exp(-2yp)\,d\Omega(p)=\frac{1}{4}\frac{\partial^2 G}{\partial y^\alpha\partial y^\beta}$$

$$=4m^4y_\alpha y_\beta a(m)\varphi^{-\nu-2}K_{\nu+2}(\varphi)$$
$$-m^2g_{\alpha\beta}a(m)\varphi^{-\nu-1}K_{\nu+1}(\varphi),$$

giving

$$\langle P_\alpha P_\beta\rangle=\frac{K_{\nu+2}(2\lambda m)}{K_\nu(2\lambda m)}\cdot\frac{m^2}{\lambda^2}y_\alpha y_\beta$$
$$-\frac{K_{\nu+1}(2\lambda m)}{K_\nu(2\lambda m)}\frac{m}{2\lambda}g_{\alpha\beta}. \tag{4.5}$$

Equations (4.4) and (4.5) give the expected momenta and their correlation matrix

$$C_{\alpha\beta}=\langle P_\alpha P_\beta\rangle-\langle P_\alpha\rangle\langle P_\beta\rangle. \tag{4.6}$$

To gain a rough idea of the behavior of $\langle P_\alpha\rangle$ and $C_{\alpha\beta}$, we consider the limiting cases $\lambda m\to\infty$. From (A3) and (A4) we obtain

$$\frac{\nu}{\lambda^2}y_\alpha \overset{0}{\sim} \langle P_\alpha\rangle \overset{\infty}{\sim}\frac{m}{\lambda}y_\alpha, \tag{4.7}$$

$$\frac{\nu}{\lambda^2}\left(\frac{y_\alpha y_\beta}{\lambda^2}-\tfrac{1}{2}g_{\alpha\beta}\right)\overset{0}{\sim}C_{\alpha\beta}$$

$$\overset{\infty}{\sim}\frac{m}{2\lambda}\left(1+\frac{n}{2\lambda m}\right)\left(\frac{y_\alpha y_\beta}{\lambda^2}-g_{\alpha\beta}\right)+\frac{n}{8\lambda^2}g_{\alpha\beta}. \tag{4.8}$$

Hence the uncertainties in energy and momentum obey

$$\frac{n-1}{2\lambda^2}\left(\frac{1}{2}+\frac{\mathbf{y}^2}{\lambda^2}\right)\overset{0}{\sim}C_{00}\overset{\infty}{\sim}\frac{n}{8\lambda^2}+\frac{m}{2\lambda}\frac{\mathbf{y}^2}{\lambda^2}, \tag{4.9}$$

$$\frac{n-1}{2\lambda^2}\left(\frac{1}{2}+\frac{y_k^2}{\lambda^2}\right)\overset{0}{\sim}C_{kk}\overset{\infty}{\sim}\frac{m}{2\lambda}-\frac{n}{8\lambda^2}+\frac{m}{2\lambda}\cdot\frac{y_k^2}{\lambda^2}. \tag{4.10}$$

Finally, we need an estimate on the uncertainty in position: At $x_0=0$,

$$\langle p|(X_k-x_k)e_z\rangle=\left[i\left(\frac{\partial}{\partial p_k}-\frac{p_k}{2\omega^2}\right)-x_k\right](2\pi)^{-n/2}$$
$$\times\exp(-y_0\omega+\mathbf{y}\cdot\mathbf{p}-i\mathbf{x}\cdot\mathbf{p})$$
$$=i\left(y_k-y_0\frac{p_k}{\omega}-\frac{p_k}{2\omega^2}\right)\langle p|e_z\rangle;$$

hence

$$\langle(X_k-x_k)^2\rangle=G^{-1}\int\left[y_k-y_0\frac{p_k}{\omega}\left(1+\frac{1}{2y_0\omega}\right)\right]^2$$
$$\times\exp(-2yp)\,d\Omega(p). \tag{4.11}$$

The integral is difficult to evaluate, and we merely derive an upper bound in the rest frame. Setting $\mathbf{y}=0$ and $y_0=\lambda$,

$$\langle(X_k-x_k)^2\rangle=\lambda^2G^{-1}\int\frac{p_k^2}{\omega^2}\left(1+\frac{1}{2y_0\omega}\right)^2\exp(-2\lambda\omega)\,d\Omega(p)$$

$$\le\lambda^2G^{-1}\int\left(1+\frac{1}{2y_0\omega}\right)^2\exp(-2\lambda\omega)\,d\Omega(p)$$

$$=\lambda^2+\lambda G^{-1}\int\left(\frac{1}{\omega}+\frac{1}{4\lambda\omega^2}\right)\exp(-2\lambda\omega)\,d\Omega(p).$$

Now

$$-\frac{1}{2\lambda m}\frac{\partial G}{\partial m}=\int\left(\frac{1}{\omega}+\frac{1}{2\lambda\omega^2}\right)\exp(-2\lambda\omega)\,d\Omega(p),$$

hence

$$\langle(X_k-x_k)^2\rangle\le\lambda^2-\frac{1}{2mG}\frac{\partial G}{\partial m}$$

$$=\lambda^2-\frac{2\lambda^2b(y)}{G}\frac{1}{\varphi}\frac{\partial}{\partial\varphi}(\varphi^\nu K_\nu(\varphi))$$

$$=\lambda^2+\frac{2\lambda^2b(y)}{G}\varphi^{\nu-1}K_{\nu-1}(\varphi).$$

The position uncertainty therefore satisfies

$$\langle(X_k-x_k)^2\rangle\le\lambda^2+\frac{\lambda}{m}\frac{K_{\nu-1}(2\lambda m)}{K_\nu(2\lambda m)}; \tag{4.12}$$

hence

$$\langle(X_k-x_k)^2\rangle\lesssim[\nu/(\nu-1)]\lambda^2\quad\text{as }\lambda m\to0, \tag{4.13}$$

$$\lesssim\lambda^2+\lambda/m\quad\text{as }\lambda m\to\infty. \tag{4.13'}$$

For $\nu\equiv(n-1)/2=1$ (which is in fact the physical case), (4.13) must be replaced with

$$\langle (X_k - x_k)^2 \rangle \lesssim \lambda^2 - 2\lambda^2 \ln(2\lambda m) \text{ as } \lambda m \to 0. \tag{4.13'}$$

Thus $\Delta X_k \to 0$ when $\lambda \to 0$.

We can now draw consequences from the above computations. Equations (4.2) and (4.4) confirm that e_{x-iy} is a wavepacket centered about $\mathbf{x}$ with expected energy–momentum proportional to $(y_0, \mathbf{y})$. Note that

$$(\langle P_\alpha\rangle\langle P^\alpha\rangle)^{1/2} = m\frac{K_{\nu+1}(2\lambda m)}{K_\nu(2\lambda m)} \equiv m_\lambda > m. \tag{4.14}$$

We shall call m_λ the "effective mass" for the particle in P_λ. The factor $K_{\nu+1}/K_\nu$ represents a kind of renormalization which takes into effect the fluctuations in energy–momentum. m_λ has the asymptotic behavior

$$\nu/\lambda \overset{0}{\sim} m_\lambda \overset{\infty}{\sim} m. \tag{4.15}$$

Equations (4.7)–(4.10), (4.13), and (4.15) show the following pattern: When $\lambda m \to 0$, the expectations and uncertainties of physical observables in the state e_z become independent of the mass. Thus, roughly, when $\lambda \to 0$ (i.e., z approaches the boundary of $\mathcal{T}$), analyticity fails and fluctuations take over. On the other hand, we have seen that $\lambda m \equiv \lambda mc \equiv \beta mc^2 \to \infty$ gives a smooth transition to the nonrelativistic formalism (Theorem 3). Thus we expect $\langle P_k\rangle \to my_k/\beta \equiv my_k/\lambda$, $C_{kk} \to m/2\beta \equiv m/2\lambda$, and $\langle (X_k - x_k)^2\rangle \to \beta/2m \equiv \lambda/2m$. The first two are born out by (4.7) and (4.10). Equation (4.13′), though consistent with this expectation, shows that in obtaining the estimate (4.12) we gave up too much ground.

The nonrelativistic wavepackets e_z^β have the attractive feature of being minimum-uncertainty states. So far we have not shown that the e_z have a similar property. Now uncertainty products do not seem to be natural measure of the optimality of relativistic states. The position operators X_k are not covariant,[11] and furthermore it is not obvious how to define an *invariant* counterpart to the uncertainty product. We conclude by proving that the e_z are characterized by a simple, invariant property which we propose as an adequate substitute for minimal uncertainty. For $z \in \mathcal{T}$ let

$$\tilde{e}_z(w) = \langle e_w | e_z\rangle / \|e_z\|, \quad w \in \mathcal{T}.$$

Theorem 4: Let $z \in \mathcal{T}$. Then $\tilde{e}_z$ is the unique (up to a constant phase factor) solution to the following problem: Find $f \in \mathcal{K}$ such that $\|f\| = 1$ and $|f(z)|$ is a maximum.

Proof: We have

$$|\langle e_z | f\rangle| \le \|e_z\|\|f\|,$$

and equality holds if and only if f is a constant multiple of e_z. ∎

Remark: Theorem 4 can be restated as a variational principle[19]: $\hat{e}_z(w) \equiv \langle e_w | e_z\rangle / \|e_z\|^2$ is the unique function f in $\mathcal{K}$ such that $f(z) = 1$ and $\|f\|$ is a minimum. The above form seems to be more appropriate for quantum mechanics. See also Ref. 20.

5. CONCLUSION

We have developed a formalism analogous to that of the coherent-state representation. By this analogy we have called P_λ a "phase space." We then showed that, at least so far as the e_z are concerned, P_λ is indeed a space parametrized by coordinates and momenta. Now in the classical notion of phase space, a central role is played by Poisson brackets and canonical transformations, i.e., by symplectic structure.[21,22] These geometrical aspects will be dealt with in a later paper, where the present formalism will be given a geometrical foundation and made manifestly covariant.

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APPENDIX A

We collect here some properties of the modified Bessel functions K_ν and evaluate some integrals needed in Secs. 3 and 4.

The functions $K_\nu(\xi)$ are defined[16] for $\mathrm{Re}\,\nu > -\frac{1}{2}$ and $\mathrm{Re}\,\xi > 0$ by

$$K_\nu(\xi) = \frac{\sqrt{\pi}(\xi/2)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty \exp(-\xi\cosh t)\sinh^{2\nu} t\, dt. \tag{A1}$$

They satisfy

$$\left(-\frac{1}{\xi}\frac{d}{d\xi}\right)^m (\xi^\nu K_\nu(\xi)) = \xi^{\nu-m}K_{\nu-m}(\xi),$$
$$\left(-\frac{1}{\xi}\frac{d}{d\xi}\right)^m (\xi^{-\nu} K_\nu(\xi)) = \xi^{-\nu-m}K_{\nu+m}(\xi), \tag{A2}$$

for $m = 1, 2, \cdots$ and

$$K_\nu(\xi) \sim \tfrac{1}{2}\Gamma(\nu)(\xi/2)^{-\nu}, \quad \xi \to 0 \quad (\nu \ne 0),$$
$$K_0(\xi) \sim -\ln(\xi/2), \quad \xi \to 0,$$
$$K_\nu(\xi) \sim \sqrt{\pi/2\xi}\, e^{-\xi}, \quad \xi \to +\infty. \tag{A3}$$

In Sec. 4 we use

$$\frac{\Gamma(\nu+k)}{\Gamma(\nu)}\left(\frac{\xi}{2}\right)^{-k} \overset{0}{\sim} \frac{K_{\nu+k}(\xi)}{K_\nu(\xi)} \overset{\infty}{\sim} 1 + \frac{k^2+2k\nu}{2\xi},$$
$$\frac{2(n-1)}{\xi^2} \overset{0}{\sim} \frac{K_{\nu+2}(\xi)}{K_\nu(\xi)} - \left(\frac{K_{\nu+1}(\xi)}{K_\nu(\xi)}\right)^2 \overset{\infty}{\sim} \frac{1}{\xi} + \frac{n}{\xi^2}. \tag{A4}$$

To evaluate

$$I(y_0, \mathbf{y}) = \int_{R^n} \frac{d^n p}{(1+\mathbf{p}^2)^{1/2}} \exp[-2y_0(1+\mathbf{p}^2)^{1/2} + 2\mathbf{y}\cdot\mathbf{p}],$$

$$\lambda \equiv (y_0^2 - \mathbf{y}^2)^{1/2} > 0,$$

note that I is Lorentz-invariant; hence

$$I(y_0, \mathbf{y}) = I(\lambda, 0) = \int_{R^n} \frac{d^n p}{(1+\mathbf{p}^2)^{1/2}} \exp[-2\lambda(1+\mathbf{p}^2)^{1/2}]$$
$$= \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \frac{r^{n-1}\,dr}{(1+r^2)^{1/2}} \exp[-2\lambda(1+r^2)^{1/2}]$$
$$= \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \sinh^{n-1} t \exp(-2\lambda\cosh t)\,dt$$
$$= 2\left(\frac{\pi}{\lambda}\right)^\nu K_\nu(2\lambda), \quad \nu \equiv \frac{n-1}{2}. \tag{A5}$$

Consequently, using (A2),

$$\int_{R^n}\frac{d^n p}{(1+\mathbf{p}^2)^{1/2}}p_\alpha \exp[-2y_0(1+\mathbf{p}^2)^{1/2}+2\mathbf{y}\cdot\mathbf{p}]$$
$$=-\frac{1}{2}\frac{\partial}{\partial y^\alpha}I(y_0,\mathbf{y})$$
$$=4y_\alpha\left(-\frac{1}{2\lambda}\frac{\partial}{\partial(2\lambda)}\right)\left[\left(\frac{\pi}{\lambda}\right)^\nu K_\nu(2\lambda)\right]$$
$$=\frac{2}{\pi}y_\alpha\left(\frac{\pi}{\lambda}\right)^{\nu+1}K_{\nu+1}(2\lambda), \qquad \text{(A6)}$$

where $p_0=(1+\mathbf{p}^2)^{1/2}$.

APPENDIX B. PROOF OF THEOREM 3

We can set $m=\beta=1$ without loss. Note

$$f_1^{\rm NR}(\mathbf{x}-i\mathbf{y})=\exp(-\mathbf{y}^2/2)\langle e^1_{\mathbf{x}-i\mathbf{y}}|f\rangle_{L^2(R^n)}.$$

Hence by (2.8) and (2.9),

$$\|f_1^{\rm NR}\|^2_{L^2(C^n)}=\pi^{n/2}\|f_1^{\rm NR}\|_{H_1}=\pi^{n/2}\|\hat f\|_{L^2(R^n)}<\infty.$$

Note also

$$\|ce^{c^2}f_c\|^2_{L^2(C^n)}=\frac{c^2e^{2c^2}}{C_c}\|f_c\|^2_{K_c}$$
$$\leq \pi^{n/2}c\|\hat f\|^2_{L^2(\Omega)}[1+O(c^{-2})]$$
$$\leq \pi^{n/2}\|\hat f\|^2_{L^2(R^n)}[1+O(c^{-2})].$$

Now

$$J=\int\int dx\,dy\,|[\{(c/\omega)\exp(c^2-yp)-\exp[-\tfrac{1}{2}(\mathbf{p}-\mathbf{y})^2]\}\hat f\,]^\vee(\mathbf{x})|^2$$
$$=\int dp\,|\hat f(\mathbf{p})|^2\int dy((c/\omega)\exp(c^2-yp)-\exp[-\tfrac{1}{2}(\mathbf{p}-\mathbf{y})^2])^2.$$

Choose α,γ such that $\frac{1}{2}<\gamma<\alpha<1$. Then $\int_{|p|>c^{1-\alpha}}dp \times|\hat f(\mathbf{p})|^2\to 0$ as $c\to\infty$; hence

$$J_1\equiv\int_{|p|>c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{R^n}dy$$
$$\times\{(c/\omega)\exp(c^2-yp)-\exp[-\tfrac{1}{2}(\mathbf{p}-\mathbf{y})^2]\}^2$$
$$\leq 4\pi^{n/2}\|\chi_c\hat f\|^2_{L^2(R^n)}\to 0 \quad\text{as } c\to\infty,$$

where $\chi_c(\mathbf{p})$ is the characteristic function of $\{|\mathbf{p}|>c^{1-\alpha}\}$.

Define θ and φ by $|\mathbf{y}|=c\sinh\theta$, $|\mathbf{p}|=c\sinh\varphi$. Then $y_0=(c^2-\mathbf{y}^2)^{1/2}=c\cosh\theta$ and $\omega=c\cosh\varphi$; hence $yp=y_0\omega-\mathbf{y}\cdot\mathbf{p}\geq c^2\cosh(\theta-\varphi)\geq c^2+(c^2/2)(\theta-\varphi)^2$. Thus for arbitrary $a\geq 0$,

$$G_a(\mathbf{p})\equiv\int_{|y|>c\sinh a}dy\,\exp(2c^2-2yp)$$
$$\leq\frac{2c^n\pi^{n/2}}{\Gamma(n/2)}\int_a^\infty\sinh^{n-1}\cosh\theta\exp[-c^2(\theta-\varphi)^2]\,d\theta$$
$$\leq\frac{2^{1-n}c^n\pi^{n/2}}{\Gamma(n/2)}\int_a^\infty\exp[(n-1)\theta](e^\theta+e^{-\theta})\exp[-c^2(\theta-\varphi)^2]d\theta$$
$$\leq\frac{2^{2-n}c^n\pi^{n/2}}{\Gamma(n/2)}\int_a^\infty\exp[n\theta-c^2(\theta-\varphi)^2]\,d\theta$$
$$=\frac{2^{2-n}c^{n-1}\pi^{n/2}}{\Gamma(n/2)}\exp(n\varphi+n^2/4c^2)\int_{c(a-\varphi)-n/2c}^\infty\exp(-u^2)\,du.$$

Let $a=\sinh^{-1}(c^{-\gamma})$. Then, for $|\mathbf{p}|<c^{1-\alpha}$,

$$c(a-\varphi)-n/2c\geq c[\sinh^{-1}(c^{-\gamma})-\sinh^{-1}(c^{-\alpha})]-n/2c$$
$$\equiv g(c).$$

$g(c)$ is independent of $\mathbf{p}$ and $g(c)\sim c^{1-\gamma}$, $c\to\infty$. Also, $|\mathbf{p}|<c^{1-\alpha}\Rightarrow\varphi<c^{-\alpha}$. Hence

$$J_2\equiv\int_{|p|<c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{|y|>c^{1-\gamma}}dy\,\exp(2c^2-2yp)$$
$$\leq\frac{2^{2-n}c^{n-1}\pi^{n/2}}{\Gamma(n/2)}\exp(nc^{-\alpha}+n^2/4c^2)\left(\int_{g(c)}^\infty\exp(-u^2)\,du\right)$$
$$\times\|\hat f\|^2_{L^2(R^n)}\to 0,\quad c\to\infty.$$

Now

$$2c^2-2yp=y^2+p^2-2yp=(y-p)^2$$
$$\equiv(y_0-\omega)^2-(\mathbf{y}-\mathbf{p})^2$$
$$\geq-(\mathbf{y}-\mathbf{p})^2.$$

Hence

$$\int_{|p|<c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{|y|>c^{1-\gamma}}dy\,\exp[-(\mathbf{y}-\mathbf{p})^2]\leq J_2$$

and

$$\int_{|p|<c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{|y|>c^{1-\gamma}}dy\{(c/\omega)\exp(c^2-yp)$$
$$-\exp[-\tfrac{1}{2}(\mathbf{y}-\mathbf{p})^2]\}^2\leq 4J_2\to 0\quad\text{as } c\to\infty.$$

Finally,

$$J_3\equiv\int_{|p|<c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{|y|<c^{1-\gamma}}dy\{(c/\omega)\exp(c^2-yp)$$
$$-\exp[-\tfrac{1}{2}(\mathbf{y}-\mathbf{p})^2]\}^2$$
$$=\int_{|p|<c^{1-\alpha}}dp\,|\hat f(\mathbf{p})|^2\int_{|y|<c^{1-\gamma}}dy\,\exp[-(\mathbf{y}-\mathbf{p})^2]$$
$$\times[(c/\omega)\exp(c^2\delta^2/2)-1]^2,$$

where

$$\delta=|(1+\mathbf{y}^2/c^2)^{1/2}-(1+\mathbf{p}^2/c^2)^{1/2}|$$
$$\leq\tfrac{1}{2}|\mathbf{y}^2/c^2-\mathbf{p}^2/c^2|\leq\tfrac{1}{2}(c^{-2\gamma}+c^{-2\alpha})\leq c^{-2\gamma}.$$

We have used the estimate

$$|(1+u^2)^{1/2}-(1+v^2)^{1/2}|$$
$$=\left|\int_u^v\frac{x\,dx}{(1+x^2)^{1/2}}\right|\leq\left|\int_u^v x\,dx\right|=\tfrac{1}{2}|v^2-u^2|.$$

Hence for sufficiently large c and $|\mathbf{p}|<c^{1-\alpha}$,

$$[(c/\omega)\exp(c^2\delta^2/2)-1]^2$$
$$\leq\exp(c^2\delta^2)+1-(2c/\omega)\exp(c^2\delta^2/2)$$
$$\leq(1+2c^2\delta^2)+1-2(1-p^2/2c^2)\exp(c^2\delta^2/2)$$
$$\leq 2[1-\exp(c^2\delta^2/2)]+2c^2\delta^2+c^{-2\alpha}\exp(c^2\delta^2/2)$$
$$\leq 2c^2\delta^2+c^{-2\alpha}(1+c^2\delta^2)$$
$$\equiv h(c)\to 0\quad\text{as } c\to\infty.$$

Thus

$$J_3 \leq h(c) \int_{|p|<c^{1-\alpha}} dp\, |\hat{f}(p)|^2 \int_{|y|<c^{1-\gamma}} dy\, \exp[-(\mathrm{y}-\mathrm{p})^2]$$

$$\leq h(c)\pi^{n/2}\|\hat{f}\|^2_{L^2(R^n)} \to 0 \quad \text{as } c \to \infty,$$

which proves that $J \to 0$ as $c \to \infty$. ■

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