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RELATIVISTIC COHERENT-STATE REPRESENTATIONS

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1. INTRODUCTION

The coherent-state representation and its variants [1-3] have found many applications in quantum physics, in particular as a tool for the study of the classical limit [4-6]. For finite degrees of freedom, such representations are usually confined to non-relativistic systems. The purpose of this paper is to construct similar representations which are applicable to relativistic particles. In section 2 we develop a family of representations for the dynamics of a free non-relativistic particle which is closely related to the coherent-state representation. This family is extended in section 3 to include relativistic particles. In section 4 we summarize some properties of the new wave packets.

2. NON-RELATIVISTIC PARTICLE

The wave function $f(\vec{x},t)$ for a non-relativistic free particle in R^n evolves under the Schrödinger equation

$$i \frac{\partial f}{\partial t} = Hf, \quad H = -\frac{1}{2m}\Delta. \tag{2.1}$$

The solutions are given by

$$f(\vec{x},t) = (e^{-itH}f)(\vec{x}) = (2\pi)^{-n/2} \int_{R^n} e^{-itp^2/2m+i\vec{x}\cdot\vec{p}}\hat{f}(\vec{p})d^np \quad (2.2)$$

where $\hat{f}(\vec{p})$ is the Fourier transform of the initial function $f(\vec{x},0)$. Now let $\vec{z} = \vec{x}-i\vec{y} \in C^n$ and let $\tau = t-i\beta$ be in the lower-half plane C^- (i.e., $\beta > 0$). Then $\exp(-i\tau p^2/2m + i\vec{z}\cdot\vec{p})$ decays rapidly as $|\vec{p}| \to \infty$ and eq. (2.2) defines a function $f(\vec{z},\tau)$ holomorphic in $\mathcal{D} = C^n \times C^-$. Let $G = \{f(\vec{z},\tau): \hat{f} \in L^2(R^n)\}$ be the vector space of all such functions. Then for each $\beta > 0$ the function $f_\beta(\vec{z},t) = f(\vec{x}-i\vec{y},t-i\beta)$ satisfies (2.1) in $\vec{x}$ and t. Let G_β be the space of all such functions $f_\beta(\vec{z},t)$. On G_β define the map $(e^{-itH}f_\beta)(\vec{z},s) = f_\beta(\vec{z},s+t)$. We are going to make G_β into a Hilbert space such that e^{-itH} is unitary for every real t, giving us a unitary representation of dynamics on G_β for every $\beta > 0$. Although these representations are all unitarily equivalent, the spaces G_β have some interesting properties, as we shall see.

Thus let $\beta > 0$ and $\vec{z} = \vec{x}-i\vec{y} \in C^n$. Then

$$f_\beta(\vec{z},0) = (e^{-\beta H}f)(\vec{z}) = (2\pi)^{-n/2} \int_{R^n} e^{-\beta p^2/2m+i\vec{z}\cdot\vec{p}}\hat{f}(\vec{p})d^np$$
$$\equiv \langle e^\beta_{\vec{z}}|f\rangle \quad (2.3)$$

where

$$\langle e^\beta_{\vec{z}}|p\rangle = (2\pi)^{-n/2}\exp(-\beta p^2/2m+i\vec{z}\cdot\vec{p}) \quad (2.4)$$

with Fourier transform

$$\langle e^\beta_{\vec{z}}|\vec{x}'\rangle = \left(\frac{2\pi\beta}{m}\right)^{-n/2}\exp\left(-\frac{m(\vec{z}-\vec{x}')^2}{2\beta}\right). \quad (2.5)$$

The e^β_z are minimum-uncertainty spherical wave packets with $\langle X_k\rangle = x_k = \mathrm{Re}(z_k)$, $\langle P_k\rangle = (m/\beta)y_k$ and diameter $\Delta X_k = \sqrt{\beta/2m}$ $(k = 1,2,\ldots,n)$. For f_β in G_β define

$$\|f\|_\beta^2 = \int_{C^n} |f_\beta(\vec{z},0)|^2 d\mu_\beta(\vec{z}), \tag{2.6}$$

where

$$d\mu_\beta(\vec{z}) = (\frac{m}{\pi\beta})^{n/2}\exp(-\frac{my^2}{\beta})d^nx d^ny. \tag{2.7}$$

Theorem 1. Let $t \in R$, $\beta > 0$, $f \in L^2(R^n)$ and $f_\beta = e^{-\beta H}f$. Then

$$\|f\|_\beta = \|f\|. \tag{2.8}$$

In particular,

(a) $\|\cdot\|_\beta$ is a norm on G_β under which G_β is a Hilbert space.

(b) The map $e^{-\beta H}$ is unitary from $L^2(R^n)$ onto G_β.

(c) The map e^{-itH} is unitary on G_β.

Remarks. 1. (2.8) can of course be polarized to give a resolution of the identity: for f, g in $L^2(R^n)$,

$$\langle f|g\rangle_\beta \equiv \int_{C^n} \langle f|e^\beta_{\vec{z}}\rangle\langle e^\beta_{\vec{z}}|g\rangle\, d\mu_\beta(\vec{z}) = \langle f|g\rangle. \tag{2.9}$$

2. $e^{-\beta H}$ intertwines [7] the dynamics on $L^2(R^n)$ with the dynamics on G_β.

Proof. Let $f \in S(R^n)$. By (2.3), $f_\beta(\vec{x}-i\vec{y},0) = \check{g}_{\beta,\vec{y}}(\vec{x})$ where $g_{\beta,\vec{y}}(\vec{p}) = \exp(-\beta p^2/2m + \vec{y}\cdot\vec{p})\ \hat{f}(\vec{p})$ and $\check{g}$ denotes the inverse Fourier transform of g. Thus by Plancherel's theorem (and Fubini's),

$$\|f\|_\beta^2 = (\frac{m}{\pi\beta})^{n/2} \int_{R^n} e^{-my^2/\beta}d^ny \int_{R^n} d^np\ e^{-\beta p^2/m+2\vec{y}\cdot\vec{p}}|\hat{f}(\vec{p})|^2$$

$$= \int_{R^n} |\hat{f}(\vec{p})|^2 d^np = \|f\|^2,$$

which proves (2.8) for f in $S(R^n)$, hence also in $L^2(R^n)$ by continuity. (a)-(c) are obvious.

3. RELATIVISTIC PARTICLE

We sketch a generalization of the results of section 2 to relativistic particles. We confine ourselves to n=3.

The evolution of a free scalar relativistic particle of mass m > 0 is given by the Klein-Gordon equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta - m^2\right)f(\vec{x},t) = 0. \qquad (3.1)$$

We consider only positive-energy solutions. These are given by

$$f(\vec{x},x_o) = (e^{-ix_oH}f)(\vec{x}) = (2\pi)^{-3/2}\int_{R^3} e^{-ixp}\hat{f}(\vec{p})d\Omega(\vec{p}), \qquad (3.2)$$

where $x_o = ct$, $H = \sqrt{(mc)^2-\Delta}$, $xp = x_o\omega-\vec{x}\cdot\vec{p}$ with $\omega = \sqrt{(mc)^2+\vec{p}^2}$, and $d\Omega(p) = d^3p/\omega$ is the Lorentz-invariant measure in momentum space. $\hat{f}$ is the ordinary Fourier transform on R^3. For every $\hat{f}$ in $L^2(\Omega)$ the solution $f(\vec{x},x_o)$ is the boundary-value of a function $f(\vec{z},z_o) = f(z)$ holomorphic in the forward tube [8]

$$\mathcal{T} = \{x-iy \in C^4: x \in R^4,\ y \in V_+\},$$

where

$$V_+ = \{y \in R^4: y_o > |\vec{y}|\}$$

is the open forward light cone. This is so because

$$|e^{-izp}| = e^{-yp} < \exp(-(y_o-|\vec{y}|)|\vec{p}|)$$

decays rapidly as $|\vec{p}| \to \infty$ for fixed z in $\mathcal{T}$. $\mathcal{T}$ will replace $\mathcal{D} = C^3 \times C^-$ of section 2 and is strictly contained in $\mathcal{D}$. The analogue of $\mathcal{G}$ is the space $\mathcal{K} = \{f(\vec{z},z_o):\hat{f} \in L^2(\Omega)\}$. To obtain counterparts of the G_β we need a phase space. In section 2 that was the set $\{(\vec{z},\tau) \in \mathcal{D}: \tau = -i\beta\} \approx C^3$. This will not do since it is not contained in T. Thus we deform it: let

$$P_\lambda = \{z = x-iy \in C^4: z_0 = -i\sqrt{\lambda^2+\vec{y}^2}\}, \quad \lambda \geq 0.$$

The functions

$$f_\lambda(\vec{z},x_0) = f(\vec{x}-i\vec{y},x_0-i\sqrt{\lambda^2+\vec{y}^2})$$

satisfy (3.1) in $\vec{x}$ and x_0 = ct. Let $K_\lambda = \{f_\lambda(\vec{z},x_0): \hat{f} \in L^2(\Omega)\}$ and denote the map $\hat{f}(\vec{p}) \to f_\lambda(\vec{z},x_0)$ by U_λ. Define dynamics on K_λ by

$$(e^{-ix_0'H}f_\lambda)(\vec{z},x_0) = f_\lambda(\vec{z},x_0+x_0').$$

For $\lambda > 0$,

$$\begin{aligned} f_\lambda(\vec{z},0) &= (2\pi)^{-3/2}\int \exp(-\sqrt{\lambda^2+\vec{y}^2}\,\omega+i\vec{z}\cdot\vec{p})\hat{f}(\vec{p})d\Omega(\vec{p}) \\ &\equiv \langle e_{\vec{z}}^\lambda | f\rangle \end{aligned} \tag{3.3}$$

where

$$\langle e_{\vec{z}}^\lambda | p\rangle = (2\pi)^{-3/2}\exp(-\sqrt{\lambda^2+\vec{y}^2}\,\omega+i\vec{z}\cdot\vec{p}) \tag{3.4}$$

and all inner products are in $L^2(\Omega)$ until further notice. The e_z^λ are in $L^2(\Omega)$: for $z = x-iy$ in P_λ and $z' = x'-iy'$ in $P_{\lambda'}$ (where $\lambda,\lambda' > 0$),

$$\begin{aligned} \langle e_{\vec{z}}^\lambda | e_{\vec{z}'}^{\lambda'}\rangle &= (2\pi)^{-3}\int \exp\{-(y_0+y_0')\omega+i(\vec{z}-\bar{\vec{z}}')\cdot\vec{p}\}d\Omega(\vec{p}) \\ &= -2i\Delta_+(z-\bar{z}') \\ &= \frac{mc}{4\pi^2\eta}K_1(2\eta mc), \end{aligned} \tag{3.5}$$

where y_0 denotes $\sqrt{\lambda^2+\vec{y}^2}$, Δ_+ is the two-point function for the free scalar field of mass m [8] and $2\eta = [-(z-\bar{z}')^2]^{\frac{1}{2}}$ is defined by analytic continuation from $[-(z-\bar{z})^2]^{\frac{1}{2}} = [4y^2]^{\frac{1}{2}} = 2\lambda$ for $z = z' = x-iy$ in P_λ. K_n $(n = 0,1,2,\ldots)$ denotes a modified Bessel function. For $\lambda=0$, (3.3) still gives $f_o(\vec{z},0)$ and the functions $e^o_{\vec{z}}$ are still defined, but are no longer in $L^2(\Omega)$, as (3.5) shows. For $f_\lambda \in K_\lambda$ $(\lambda \geq 0)$ define

$$\|f\|^2_\lambda = \int_{C^3} |f_\lambda(\vec{z},0)|^2 d\mu_\lambda(\vec{z}) \tag{3.6}$$

where

$$d\mu_\lambda(\vec{z}) = C_\lambda d^3x d^3y \tag{3.7}$$

with $C_\lambda = [2\pi(\lambda/mc)^2 K_2(2\lambda mc)]^{-1}$ for $\lambda > 0$ and $C_o = (mc)^4/\pi$. Then our main result is the following

Theorem 2. Let $\lambda \geq 0$ and $\hat{f} \in L^2(\Omega)$. Then

$$\|f\|_\lambda = \|\hat{f}\|. \tag{3.8}$$

In particular,

(a) $\|\cdot\|_\lambda$ is a Lorentz-invariant norm on K_λ under which K_λ is a Hilbert space.

(b) The map U_λ is unitary from $L^2(\Omega)$ onto K_λ.

(c) e^{-ix_oH} is unitary on K_λ.

The remarks following Theorem 1 apply here as well.

Comparing the measures (2.7) and (3.7), note that $d\mu_\lambda$ has no weight function. This is a consequence of the curvature of the phase space P_λ. The "weight" has been absorbed into the functions f_λ themselves, which are consequently bounded:

$$|f_\lambda(\vec{z},0)|^2 = |\langle e^\lambda_{\vec{z}}|f\rangle|^2 \leq \|e^\lambda_{\vec{z}}\|^2\|\hat{f}\|^2 = \frac{mc}{4\pi^2\lambda} K_1(2\lambda mc)\|\hat{f}\|^2. \tag{3.9}$$

Finally note that Theorem 2 gives us a unitary, irreducible representation of the restricted Poincaré group on K_λ. Define the action on K by $(U(g)f)(z) = f(g^{-1}z)$, $g \in P^{\uparrow}_{+}$. This induces an action on K_λ with the desired properties.

4. CONCLUSION

The $e^{\lambda}_{\vec{z}}$ have other interesting properties which we can only mention here for lack of space. In the state $e^{\lambda}_{\vec{z}}$, the particle appears as a wave packet centered about $\vec{x} = \mathrm{Re}(\vec{z})$ with expected momentum proportional to $\vec{y} = -\mathrm{Im}(\vec{z})$. The wave packet, which is spherical in the rest frame, shows contraction in the direction of motion and has minimal undertainties in a natural sense. It diameter increases from zero (when $\lambda mc \to 0$) to $\sim \sqrt{\lambda/2mc}$ (when $\lambda mc \to \infty$). Thus $e^{\lambda}_{\vec{z}}$ describes an extended, relativistic particle.

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